Excellence in
Mathematics

SENIOR SECONDARY
3
TEACHER'S GUIDE

Reviewer and Contributor
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Introduction

The purpose of the curriculum

The main objectives of the Mathematics curriculum are to prepare the students to:

- acquire the mathematical literacy necessary to function in an information age
- cultivate the understanding and application of mathematical concepts and skills necessary to thrive in the ever-changing technological world
- develop the essential skills of problem-solving, communication, reasoning and connection in the study of Mathematics
- take advantage of the numerous career opportunities provided by Mathematics
- prepare for further studies in Mathematics and other related fields.

The role of the teacher

One of the principal duties of a Mathematics teacher is to prepare and present good lessons to students. In order to do this, the teacher needs to:

- be as well informed as possible on the scheme of work of the subject
- know the aims and objective of each topic
- select appropriate content material
- decide on the best methods of presentation such as group work, worksheets, question–answer sessions and debate
- remain informed about social and environmental issues and other current news in Nigeria and the rest of the world
- through innovative teaching approaches encourage learning that will promote creativity and critical thinking.

To be effective in presentation, the teacher should prepare a written or typed plan for each lesson. This must include aims, objectives, resources, time frames, content for the lesson, activities, homework, assessment and ideas for additional worksheets to cater for students who require extension or learning support (remediation).

Prepare each topic in advance. It is the Mathematics teacher's responsibility to involve the students actively in the learning process. It is a proven fact that students learn far more by *doing* than by *listening*.

Mathematics involves being curious and asking questions. Where possible, teachers should ask questions to engage the students, encourage independent thought processes, and develop problem-solving skills. Teaches should start their lessons by asking the students to give the answers to about five questions that are related to the lesson. This will settle students into the lesson.

Teachers can use different types of questions in their lessons:
- diagnostic questions enable teachers to determine prior knowledge on the topic
- consolidation questions help students learn challenging concepts
- questions can stimulate interest in a topic
- questions can be used to conclude a lesson.

Concluding questions will help teachers find out how well students have understood the concepts and terminology of the lesson. It will also highlight any areas that they need to revise at home or the teacher needs to revisit in the next lesson.

Teachers should ensure that they do not appear to have favourites in the class. They should ensure that they ask questions fairly and do not embarrass students who struggle to answer questions.

How to use the scheme of work

A scheme of work is defined as the part of the curriculum that a teacher will be required to teach in any particular subject. Its primary function is to provide an outline of the subject matter and its content and to indicate how much work a student should cover in any particular class. A scheme of work allows teachers to clarify their thinking about a subject and to plan and develop particular curriculum experiences that they believe may require more time and attention when preparing lessons. The criteria all teachers should bear in mind when planning a scheme of work are continuity in learning and progression of experience. You can add your own notes to the scheme of work provided on pages vi to viii.

The scheme of work is sequential. The sequence of the scheme of work is aligned with the Student's Book. Teachers should not be tempted to jump around, but rather spend time planning each term carefully to ensure that they adhere to the scheme of work and prepare students for the final examinations.

Students will cover the work for SS3 in the first two terms and use Term 3 for revision (Topic 13) and examination preparation. This time frame may differ depending on the planning at a particular school.

Each teacher's management of his or her class will have an enormous influence on their ability to adhere to the time frames. Teachers should focus on effective discipline strategies. They will have fewer issues with discipline if they are punctual, well prepared, follow a plan (which they write on the board at the start of each lesson), keep their word (do not make empty threats) and adhere to rules consistently.

A teacher of Mathematics is a professional instructor who facilitates, promotes and influences students to achieve the outcomes of the scheme of work. It is the wish of the authors that the students will, at the end of each course in the series, attain a level of Mathematics proficiency that will equip them for future studies in this field.

Scheme of work

Term 1

Topic	Lesson objectives Students should be able to:	Student's Book pages
1. Revision of SS2 work	• Recall previous lessons of SS2. • State the basic law of logarithms. • Use the logarithms table for calculations.	1–18
2. Indices and logarithms	• Show the basic laws of logarithms. • Revise the use of logarithm tables for calculations.	19–31
3. Surds	• Differentiate between rational and irrational numbers. • State the rules of addition and subtraction and apply them to simplifying surds. • State the rules of multiplication and division and apply them to simplifying surds. • Conjugate binomial surds. • Apply the concept of surds to problems that involve the trigonometric ratios of angles 30°, 60° and 45°.	32–41
4. Trigonometry	• Draw graphs on sine and cosine for angles $0 \leq x \leq 360$. • Interpret and read graphs of trigonometric ratios.	42–55
5. Matrices and determinants	• Define a matrix. • State the order and notation of a matrix. • Mention and define types of matrices. • Perform addition and subtraction on matrices. • Multiply a matrix by a scalar quantity and multiply two matrices (A and B). • Find the transpose of a matrix by interchanging the rows and columns. • Calculate the determinant of 2 by 2 matrix. • Use applications to solve simultaneous equations.	56–75
6. Simultaneous equations	• Solve simultaneous linear and quadratic equations. • Solve word problems on linear, quadratic and simultaneous linear and quadratic equations. • Solve problems on linear equations that involve capital markets. • Carry out graphical solutions of simultaneous linear equations and trigonometric equations.	76–86

Topic	Lesson objectives Students should be able to:	Student's Book pages
7. Surface area and volume of spheres	• Find the surface area of a sphere. • Find the volume of a sphere.	87–106
8. Longitude and latitude	• Describe the earth as a sphere. • Identify (using skeletal globe) and locate (on a globe) and appreciate the following: – north and south – poles – lines of longitude – lines of latitude – the meridian and the equator – parallels of latitude – the radius of the parallel of latitude – the radius of the earth. • Recall, state the formula and solve problems on: – the arc length of a curve. • Solve problems on a longitude and latitude.	107–124
9. Finance	• Calculate simple interest given the principle, rate and time. • Be able to calculate compound interest using the formula. • Determine the depreciation value of an item. • Compute the annuities of a given problem. • Compute the amortisation in a given problem. • Solve further problems in a capital market using logarithm table.	125–142

Term 2

Topic	Lesson objectives Students should be able to:	Student's Book pages
10. Coordinate geometry	• Identify the Cartesian rectangular coordinate. • Draw and interpret linear graphs. • Determine the distance between two coordinate points. • Find the midpoint of the line that joins two points. • Apply the concept to real-life situations. • Define and determine the gradient and intercepts of a line. • Determine the equation of a line. • Find the angle between two intersecting straight lines. • Apply linear graphs to real-life situations.	143–166

Topic	Lesson objectives Students should be able to:	Student's Book pages
11. Differentiation of algebraic functions	• Explain the meaning of differentiation. • Differentiate from the first principle. • Recognise the standard derivatives of some basic functions. • Apply the rules of differentiation of functions. • Apply differentiation to capital markets and real-life situations.	167–185
12. Integration of simple algebraic functions	• Recognise integration as the reverse of differentiation. • Recognise a few standard integrals of polyamides and algebraic functions. • Apply techniques of integration such as: – integration by substitution – integration by parts – integration by partial fractions. • Apply integration to real-life situations.	186–205

Term 3

Topic	Lesson objectives	Student's Book pages
13. Revision	Theme 1 Numbers and numeration Theme 2 Algebraic processes Theme 3 Geometry Theme 4 Statistics Theme 5 Introductory calculus	206–252
Sample examination papers	Paper 1 Paper 2	253–261 262–266

TOPIC 1 Revision

Introduction to this topic

This first topic is a revision of the main concepts covered in Mathematics in SS1 and SS2. We have not attempted to cover all the work that was done in SS1 and SS2; the focus is on the concepts that are revisited in SS3. The material is arranged by theme and follows a logical progression through the curriculum.

Preparation for this topic

Prepare posters that show the main formulae that are used in this topic. Display these posters in your classroom so that students can refer to them easily. Before presenting this topic to your class, decide what the best method will be to work through this revision material with your class. Use the exercises as you think they will work best. In some cases, you might choose to ask your students to answer only a few selected questions or you may decide to skip a whole exercise. In other cases, you might prefer to postpone certain sections during revision and return to them later in the year, if you think your students will benefit more by using the revision as an introduction to this year's work.

Introducing students to the material

Explain to your class how you have decided to work through this revision material. You can discuss certain sections with the class, give other sections to students to complete for homework and ask students to work in pairs or in groups on other sections. Varying your methodology will help students focus on the material.

Answers

Exercise 1.1 (page 2)

1. a) $2.45 \times 10^6 = 2\,450\,000$
   b) $7.013 \times 10^{-5} = 0.00007013$
   c) $8.3137 \times 10^9 = 8\,313\,700\,000$
   d) $6.48005 \times 10^{-7} = 0.000000648005$

2. a) $9\,358 = 9.358 \times 10^3$
   b) $0.00294 = 2.94 \times 10^{-3}$
   c) $135\,290\,000\,000 = 1.3529 \times 10^{11}$
   d) $0.000000001836 = 1.836 \times 10^{-9}$

Exercise 1.2 (page 3)

1. a) $t^5 \times t = t^6$
   b) $p^9 \div p^3 = p^6$
   c) $(ab^6c^7)^0 = 1$
   d) $(-1)^5(vw^{-4})^2 = -v^2w^{-8} = -\frac{v^2}{w^8}$
   e) $(x \times y^{-5})^{-1} = x^{-1}y^5 = \frac{y^5}{x}$
   f) $a^7 \times a^{-7} = a^0 = 1$
   g) $625^{\frac{1}{4}} = (5^4)^{\frac{1}{4}} = 5$
   h) $(8y^9)^{\frac{2}{3}} = (2^3y^9)^{\frac{2}{3}} = 2^2y^6 = 4y^6$
   i) $8k^0 \times (8k)^0 \times (2k)^8 \div 64(-2k)^3$
   $= 8(1) \times 1 \times 256k \div 64(-8k^3)$
   $= -\frac{4k^8}{k^3}$
   $= -k^5$

2. a) $\sqrt[3]{m} = m^{\frac{1}{3}}$
   b) $\sqrt[4]{x^5} = x^{\frac{5}{4}}$
   c) $\sqrt{2y^3} = (2y^3)^{\frac{1}{2}}$
   d) $\sqrt[a]{b^c} = b^{\frac{c}{a}}$

3. a) $k^{\frac{1}{4}} = \sqrt[4]{k}$
   b) $(2ab)^{\frac{1}{2}} = \sqrt{2ab}$
   c) $x^{\frac{3}{5}} = \sqrt[5]{x^3}$
   d) $p^{\frac{m}{n}} = \sqrt[n]{p^m}$

Exercise 1.3 (page 4)

1. a) $\frac{3}{4}$ is rational, because it is in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
   b) -0.85 is rational, because it terminates.
   c) π is irrational, because it does not terminate and it also does not recur.

d) $\sqrt{\frac{25}{16}} = \frac{5}{4}$, which is rational, because it is in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
e) 0 is rational, because it terminates.
f) $1.\dot{3}$ is rational, because it recurs.
g) $-\sqrt{12}$ is irrational, because it does not terminate and it does not recur.
h) $\sqrt[3]{-8} = -2$, which is rational, because it terminates.

2. **a)** x lies exactly halfway between $\frac{a}{b}$ and $\frac{c}{d}$
$\therefore x = \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{1}{2}\left(\frac{ad + bc}{bd}\right) = \frac{ad + bc}{2bd}$
b) x is a rational number, because it is in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
c) Between any two rational numbers such as r and t, there is another rational number (s) that lies exactly halfway between the two numbers. The same applies to r and s and to s and t, and so on. This means that the set of rational numbers is infinite.

Exercise 1.4 (page 6)

1. **a)** $22 - 13 = 9$; $31 - 22 = 9$; $40 - 31 = 9$
This is an arithmetic progression. The common difference is 9. First term: 13
b) This is neither arithmetic nor geometric.
c) $\frac{100}{200} = \frac{1}{2}$; $\frac{50}{100} = \frac{1}{2}$; $\frac{25}{50} = \frac{1}{2}$
This is a geometric progression. The common ratio is $\frac{1}{2}$. First term: 200
d) $-2 - 6 = -8$; $-10 - (-2) = -8$; $-18 - (-10) = -8$
This is an arithmetic progression. The common difference is -8. First term: 6
e) This is neither arithmetic nor geometric.
f) $\frac{-3}{1} = -3$; $\frac{9}{-3} = -3$; $\frac{-27}{9} = -3$
This is a geometric progression. The common ratio is -3. First term: 1

2. **a)** $a = 13$ and $d = 9$
$$\begin{aligned}\therefore T_n &= a + (n - 1)d \\ &= 13 + (n - 1)(9) \\ &= 13 + 9n - 9 \\ &= 9n + 4\end{aligned}$$
c) $a = 200$ and $r = \frac{1}{2}$
$\therefore T_n = ar^{n-1} = 200 \times \left(\frac{1}{2}\right)^{n-1} = 200 \times 2^{1-n}$

d) $a = 6$ and $d = -8$

$\therefore T_n = a + (n - 1)d$

$= 6 + (n - 1)(-8)$

$= 6 - 8n + 8$

$= -8\text{n} + 14$

f) $a = 1$ and $r = -3$

$\therefore T_n = ar^{n-1} = 1(-3)^{n-1} = (-3)^{n-1}$

3. **a)** $T_n = \text{a}r^{n-1} = 3(2)^{n-1}$

b) $T_n = 24$

$3(2)^{n-1} = 24$

$\therefore 2^{n-1} = 8$

$= 2^3$

$\therefore n - 1 = 3$

$\therefore n = 4$

So, $T_4 = 24$

c) $T_n = 3(2)^{n-1}$

$\therefore T_{24} = 3 \times 2^{24-1} = 3 \times 2^{23} = 25\ 165\ 824$

4. $S_n = \frac{n}{2}[2a + (n - 1)d]$

$S_8 = 116$

$4(2a + 7\text{d}) = 116$

$\therefore 2a + 7d = 29$ ①

$S_{18} = -9$

$9(2a + 17d) = -9$

$\therefore 2a + 17d = -1$ ②

② − ①:

$10d = -30$

$\therefore d = -3$ ③

Substitute equation ③ into equation ①:

$2a + 7(-3) = 29$

$\therefore 2a - 21 = 29$

$\therefore 2a = 50$

$\therefore a = 25$

So, $a = 25$ and $d = -3$.

5. **a)** This geometric progression converges because $-1 < 0.3 < 1$.

b) $S_\infty = \frac{a}{1-r} = \frac{10}{1-0.3} = 14.29$ (to two decimal places)

Exercise 1.5 (page 7)

1. a) $SI = \frac{PRT}{100} = \frac{700\,000 \times 8.4 \times 3}{100} = ₦176\,400$

   Interest he will earn: ₦176 400

   b) $A = P\left(1 + \frac{R}{100}\right)^n = 700\,000\left(1 + \frac{8.1}{100}\right)^3 = ₦884\,250.11$

   Interest = A − P = ₦884 250.11 − ₦700 000

   = ₦184 250.11

   Interest earned: ₦184 250.11

   c) Bank B will pay Dubem more interest.

2. Bank A will charge Dubem less interest.

3. $$A = P\left(1 - \frac{R}{100}\right)^n$$
$$\therefore \frac{P}{2} = P\left(1 - \frac{R}{100}\right)^n$$
$$\therefore \left(1 - \frac{R}{100}\right)^n = \frac{1}{2}$$
$$\therefore \left(1 - \frac{R}{100}\right)^5 = \frac{1}{2}$$
$$\therefore 1 - \frac{R}{100} = \sqrt[5]{\frac{1}{2}}$$
$$\therefore \frac{R}{100} = 1 - \sqrt[5]{\frac{1}{2}}$$
$$\therefore R = 100\left(1 - \sqrt[5]{\frac{1}{2}}\right)$$
$$= 12.94\% \text{ (to two decimal places)}$$

4. $A = P\left(1 + \frac{R}{100}\right)^n = 300\left(1 + \frac{7.9}{100}\right)^4 = ₦406.64$

Exercise 1.6 (page 9)

1. $$S = \frac{n}{2}[2a + (n - 1)d]$$
$$\therefore 2a + (n - 1)d = \frac{2S}{n}$$
$$\therefore (n - 1)d = \frac{2S}{n} - 2a$$
$$\therefore d = \left(\frac{2S}{n} - \frac{2a}{n} - 1\right)$$

2. $$S_\infty = \frac{a}{1 - r}$$
$$\therefore S_\infty (1 - r) = a$$
$$\therefore 1 - r = \frac{a}{S_\infty}$$
$$\therefore -r = \frac{a}{S_\infty} - 1$$
$$\therefore r = 1 - \frac{a}{S_\infty}$$

3. $$A = P(1 - \frac{r}{100})^n$$
$$\therefore \frac{A}{P} = (1 - \frac{r}{100})^n$$
$$\therefore \sqrt[n]{\frac{A}{P}} = 1 - \frac{r}{100}$$

$$\therefore \tfrac{r}{100} = 1 - \sqrt[n]{\tfrac{A}{P}}$$
$$\therefore r = 100\left(1 - \sqrt[n]{\tfrac{A}{P}}\right)$$

4. $x^2 + y^2 = r^2$
$$\therefore y^2 = r^2 - x^2$$
$$\therefore y = \pm\sqrt{r^2 - x^2}$$

5. $V = \tfrac{4}{3}\pi r^3$
$$\therefore \tfrac{4}{3}\pi r^3 = V$$
$$\therefore \pi r^3 = \tfrac{3V}{4}$$
$$\therefore r^3 = \tfrac{3V}{4\pi}$$
$$\therefore r = \sqrt[3]{\tfrac{3V}{4\pi}}$$

6. $A = P(1 + \tfrac{r}{100})^n$
$$\therefore P(1 + \tfrac{r}{100})^n = A$$
$$\therefore (1 + \tfrac{r}{100})^n = \tfrac{A}{P}$$
$$\therefore \log(1 + \tfrac{r}{100})^n = \log \tfrac{A}{P}$$
$$\therefore n \log(1 + \tfrac{r}{100}) = \log \tfrac{A}{P}$$
$$\therefore n = \frac{\log \tfrac{A}{P}}{\log(1 + \tfrac{r}{100})}$$

7. $T_n = ar^{n-1}$
$$\therefore ar^{n-1} = T_n$$
$$\therefore \log ar^{n-1} = \log T_n$$
$$\therefore (n - 1)\log ar = \log T_n$$
$$\therefore n - 1 = \frac{\log T_n}{\log ar}$$
$$\therefore n = \frac{\log T_n}{\log ar} + 1$$

Exercise 1.7 (page 9)

1. $4(2x - 9) + 6 = 10 - (12 - x)$
$$\therefore 8x - 36 + 6 = 10 - 12 + x$$
$$\therefore 7x = 28$$
$$\therefore x = 4$$

2. $\frac{3x - 8}{3} = \frac{5x + 7}{6} - 5$ Multiply all terms by the LCM of 6.
$$\therefore 2(3x - 8) = 5x + 7 - 30$$
$$\therefore 6x - 16 = 5x + 7 - 30$$
$$\therefore x = -7$$

3. $3 + \frac{4x+1}{4} = \frac{8+5x}{5}$ Multiply all terms by the LCM of 20.

$\therefore 60 + 5(4x + 1) = 4(8 + 5x)$

$\therefore 60 + 20x + 5 = 32 + 20x$

$\therefore 0 = -33$

So, there is no solution.

4. $\frac{5}{x} + 12 - \frac{8}{3x} = \frac{2x-3}{3x}$ Multiply all terms by the LCM of $3x$.

$\therefore 15 + 36x - 8 = 2x - 3$

$\therefore 34x = -10$

$\therefore x = -\frac{10}{34}$

$\therefore x = -\frac{5}{17}$

5. Cross-multiply. This has the same effect as multiplying all the terms by the LCM of $(x - 5)(x + 2)$.

$\frac{x+8}{x-5} = \frac{x-8}{x+2}$

$\therefore (x + 8)(x + 2) = (x - 8)(x - 5)$

$\therefore x^2 + 10x + 16 = x^2 - 13x + 40$

$\therefore 23x = 24$

$\therefore x = \frac{24}{23}$

6. Cross-multiply. This has the same effect as multiplying all the terms by the LCM of $(x - 3)(x + 6)$.

$\frac{x+4}{x-3} = \frac{x-3}{x+6}$

$\therefore (x + 4)(x + 6) = (x - 3)(x - 3)$

$\therefore x^2 + 10x + 24 = x^2 - 6x + 9$

$\therefore 16x = -15$

$\therefore x = -\frac{15}{16}$

Exercise 1.8 (page 11)

1. **a)** Take out the HCF of $5a$.

$5a^2 + 25ab - 10a = 5a(a + 5b - 2)$

b) Do a switch-around and take out the HCF of $2(q - 7)$.

$2p(q - 7) + 2r(7 - q) = 2p(q - 7) - 2r(q - 7)$

$= 2(q - 7)(p - r)$

c) Take out the HCF of 8 and factorise the difference of squares twice.

$128x^4 - 8 = 8(16x^4 - 1)$

$= 8(4x^2 + 1)(4x^2 - 1)$

$= 8(4x^2 + 1)(2x + 1)(2x - 1)$

d) Factorise the quadratic trinomial.

$x^2 - 2x - 24 = (x - 6)(x + 4)$

e) Factorise the quadratic trinomial.
$9x^2 + 27x + 20 = (3x + 4)(3x + 5)$

f) Take out the HCF of 2 and factorise the quadratic trinomial.
$6x^2 + 2x - 28 = 2(3x^2 + x - 14)$
$= 2(3x + 7)(x - 2)$

g) Group the terms and take out a HCF.
$8ac - 10ad + 12bc - 15bd$
$= 2a(4c - 5d) + 3b(4c - 5d)$
$= (4c - 5d)(2a + 3b)$

h) Group the terms, factorise the quadratic trinomial and then factorise the difference of squares.
$12xy + 4x^2 - 4 + 9y^2 = 4x^2 + 12xy + 9y^2 - 4$
$= (2x + 3y)(2x + 3y) - 4$
$= (2x + 3y)^2 - 4$
$= (2x + 3y + 2)(2x + 3y - 2)$

2,3. a) $x^2 + 4x + 4 = (x + 2)(x + 2) = (x + 2)^2$
b) $x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$
c) $x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)\left(x + \frac{3}{2}\right) = \left(x + \frac{3}{2}\right)^2$

4. Factorise the difference of squares and simplify the result.
$901^2 - 899^2 = (901 + 899)(901 - 899) = 1\,800(2) = 3\,600$

Exercise 1.9 (page 13)

1. a) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$, $b = -1$ and $c = 72$
$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(72)}}{2(1)}$
$= \frac{1}{2} \pm \sqrt{-287}$

b) $6x^2 - x - 4 = 0$
$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 6$, $b = -1$ and $c = -4$
$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-4)}}{2(6)}$
$= \frac{1 \pm \sqrt{1 + 96}}{12}$
$= \frac{1 \pm \sqrt{97}}{12}$
$\therefore x = 0.90$ or $x = -0.74$

c) $4x^2 + 27x - 7 = 0$
$\therefore (4x - 1)(x + 7) = 0$
$\therefore x = \frac{1}{4}$ or $x = -7$

d) $x(x + 1) = 10$

$\therefore x^2 + x^2 - 10 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$, $b = 1$ and $c = -10$

$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10)}}{2(1)}$

$= \frac{-1 \pm \sqrt{1 + 40}}{2}$

$= \frac{-1 \pm \sqrt{41}]}{2}$

$\therefore x = 2.70$ or $x = -3.70$

2. **a)** $x^2 + 4x - 12 = 0$

$\therefore x^2 + 4x = 12$

$\therefore x^2 + 4x + 2^2 = 12 + 2^2$

$\therefore (x + 2)^2 = 16$

$\therefore x + 2 = \pm \sqrt{16}$

$\therefore x = -2 \pm 4$

$\therefore x = 2$ or $x = -6$

b) $x^2 = 8(x - 1)$

$\therefore x^2 = 8x - 8$

$\therefore x^2 - 8x = -8$

$\therefore x^2 - 8x + (-4)^2 = -8 + (-4)^2$

$\therefore (x - 4)^2 = -8 + 16$

$\therefore (x - 4)^2 = 8$

$\therefore x - 4 = \pm \sqrt{8}$

$\therefore x = 4 \pm \sqrt{8}$

$\therefore x = 6.83$ or $x = 1.17$

c) $2x^2 + 12x - 16 = 0$

$\therefore x^2 + 6x - 8 = 0$

$\therefore x^2 + 6x = 8$

$\therefore x^2 + 6x + 3^2 = 8 + 3^2$

$\therefore (x + 3)^2 = 17$

$\therefore x + 3 = \pm \sqrt{17}$

$\therefore x = -3 \pm \sqrt{17}$

$\therefore x = 1.12$ or $x = -7.12$

d) $-3x^2 + 6x + 1 = 0$

$\therefore x^2 - 2x - \frac{1}{3} = 0$

$\therefore x^2 - 2x = \frac{1}{3}$

$\therefore x^2 - 2x + (-1)^2 = \frac{1}{3} + (-1)^2$

$\therefore (x - 1)^2 = \frac{1}{3} + 1$

$\therefore (x - 1)^2 = \frac{4}{3}$

$\therefore x - 1 = \pm\sqrt{\tfrac{4}{3}}$

$\therefore x = 1 \pm \sqrt{\tfrac{4}{3}}$

$\therefore x = 2.15$ or $x = -0.15$

3. **a)** $k(1)^2 + (1) - 2k + 2 = 0$

$\therefore k + 1 - 2k + 2 = 0$

$\therefore -k = -3$

$\therefore k = 3$

b) $3x^2 + x - 2(3) + 2 = 0$

$\therefore 3x^2 + x - 4 = 0$

$\therefore (3x + 4)(x - 1) = 0$

$\therefore x = -\tfrac{4}{3}$ or $x = 1$

So, the other root is $-\tfrac{4}{3}$.

Exercise 1.10 (page 15)

1. **a)** $3x + 2y + 11 = 0$

$\therefore 3x + 2y = -11$ ①

$4x - 2y = 18$ ②

① + ②:

$7x = 7$

$\therefore x = 1$ ③

Substitute equation ③ into equation ②:

$4(1) - 2y = 18$

$\therefore -2y = 14$

$\therefore y = -7$

$\therefore x = 1$ and $y = -7$

b) $y + 4x = 1$

$\therefore y = 1 - 4x$ ①

$3x + y = \tfrac{7}{4}$ ②

Substitute equation ① into equation ②:

$3x + (1 - 4x) = \tfrac{7}{4}$

$\therefore 3x + 1 - 4x = \tfrac{7}{4}$

$\therefore -x = \tfrac{7}{4} - 1$

$\therefore -x = \tfrac{3}{4}$

$\therefore x = -\tfrac{3}{4}$ ③

Substitute equation ③ into equation ①:

$\therefore y = 1 - 4\left(-\tfrac{3}{4}\right)$ ①

$\therefore y = 1 + 3$

$\therefore y = 4$

$\therefore x = -\tfrac{3}{4}$ and $y = 4$

2. **a)** $2^{x+y} = 1 = 2^0$

$\therefore x + y = 0$ ①

$9^{y-x} = 3^{12}$

$\therefore (3^2)^{y-x} = 3^{12}$

$\therefore 3^{2y-2x} = 3^{12}$

$\therefore 2y - 2x = 12$

$\therefore y - x = 6$ ②

① + ②:

$2y = 6$

$\therefore y = 3$ ③

Substitute equation ③ into equation ①:

$x + 3 = 0$

$\therefore x = -3$

$\therefore x = -3$ and $y = 3$

b) $64^{x-1} = 4^{x+y}$

$\therefore (4^3)^{x-1} = 4^{x+y}$

$\therefore 4^{3x-3} = 4^{x+y}$

$\therefore 3x - 3 = x + y$

$\therefore y = 2x - 3$ ①

$3^{x-y} = \frac{1}{3}$

$\therefore 3^{x-y} = 3^{-1}$

$\therefore x - y = -1$ ②

Substitute equation ① into equation ②:

$x - (2x - 3) = -1$

$\therefore x - 2x + 3 = -1$

$\therefore -x = -4$

$\therefore x = 4$ ③

Substitute equation ③ into equation ①:

$y = 2x - 3 = 2(4) - 3 = 5$

$\therefore x = 4$ and $y = 5$

c) $25^{x+y} = 125^{4y-x}$

$\therefore (5^2)^{x+y} = (5^3)^{4y-x}$

$\therefore 5^{2x+2y} = 5^{12y-3x}$

$\therefore 2x + 2y = 12y - 3x$

$\therefore 5x = 10y$

$\therefore x = 2y$ ①

$2^{4x+y} = 8^{x+2y-1}$

$\therefore 2^{4x+y} = (2^3)^{x+2y-1}$

$\therefore 2^{4x+y} = 2^{3x+6y-3}$

$\therefore 4x + y = 3x + 6y - 3$

$\therefore x = 5y - 3$ ②

Substitute equation ② into equation ①:

$5y - 3 = 2y$

$\therefore 3y = 3$

$\therefore y = 1$ ③

Substitute equation ③ into equation ①:

$x = 2(1)$

$\therefore x = 2$

$\therefore x = 2$ and $y = 1$

3. **a)** $y = -2x - 6$ ①

$y = 3x^2 - 7x - 6$ ②

Substitute equation ② into equation ①:

$3x^2 - 7x - 6 = -2x - 6$

$\therefore 3x^2 - 7x + 2x = 0$

$\therefore 3x^2 - 5x = 0$

$\therefore x(3x - 5) = 0$

$\therefore x = 0$ or $x = \frac{5}{3}$

Substitute each value for x back into equation ① to find the corresponding values for y.

If $x = 0$, then $y = -2(0) - 6 = -6$.

If $x = \frac{5}{3}$, then $y = -2\left(\frac{5}{3}\right) - 6 = -9\left(\frac{1}{3}\right)$.

$\therefore x = 0$ and $y = -6$ or $x = \frac{5}{3}$ and $y = -9\frac{1}{3}$

b) $x + y = 7$

$\therefore y = 7 - x$ ①

$y = -x^2 + 4x + 3$ ②

Substitute equation ① into equation ②:

$7 - x = -x^2 + 4x + 3$

$\therefore x^2 - 5x + 4 = 0$

$\therefore (x - 1)(x - 4) = 0$

$\therefore x = 1$ or $x = 4$

Substitute each value for x back into equation ① to find the corresponding values for y.

If $x = 1$, then $y = 7 - 1 = 6$.

If $x = 4$, then $y = 7 - 4 = 3$.

$\therefore x = 1$ and $y = 6$ or $x = 4$ and $y = 3$

Exercise 1.11 (page 16)

1. In $\triangle$PQR and $\triangle$XYZ:

$\frac{PQ}{XY} = \frac{7.8}{4.2} = \frac{13}{7}$

$\frac{QR}{YZ} = \frac{10.4}{5.6} = \frac{13}{7}$

$\frac{PR}{XZ} = \frac{6.5}{3.5} = \frac{13}{7}$

$\therefore \triangle PQR \;|||\; \triangle XYZ$ (Sides are in the same proportion.)

2. **a)** In $\triangle ABC$ and $\triangle ADE$:

$\hat{A} = \hat{A}$ (Common)

$\hat{B}_1 = \hat{D}$ (Corresponding angles; BC ‖ DE)

$\hat{C}_1 = \hat{E}$ (Corresponding angles; BC ‖ DE)

$\therefore \triangle ABC \;|||\; \triangle ADE$ (angle, angle, angle)

b) $\frac{AB}{BD} = \frac{AC}{CE}$

$\therefore \frac{x-1}{2} = \frac{x+1}{3}$

$\therefore 3(x-1) = 2(x+1)$

$\therefore 3x - 3 = 2x + 2$

$\therefore x = 5$ cm

Exercise 1.12 (page 17)

1. **a)** $x^2 = 5^2 + 12^2$ (Pythagoras' theorem)

$= 25 + 144$

$= 169$

$\therefore x = \sqrt{169}$

$= 13$

b) $x^2 = 73^2 - 55^2$ (Pythagoras' theorem)

$= 5\,329 - 3\,025$

$= 2\,304$

$\therefore x = \sqrt{2\,304}$

$= 48$

c) $x^2 = 2^2 + 3^2$ (Pythagoras' theorem)

$= 4 + 9$

$= 13$

$x = \sqrt{13}$

d) $x^2 = 40^2 - 15^2$ (Pythagoras' theorem)

$= 1\,600 - 225$

$= 1\,375$

$\therefore x = \sqrt{1\,375}$

$= 5\sqrt{55}$

2. **a)** $\sin 30° = 0.5$ **b)** $\sin 45° = 0.71$

c) $\cos 60° = 0.5$ **d)** $\sin 120° = 0.87$

e) $\cos 135° = -0.71$ **f)** $\cos 150° = -0.87$

Exercise 1.13 (page 18)

1. **a)** $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{4 - 5}{7 - 1}$

$= -\frac{1}{6}$

b) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-2 - 3}{-2 - (-6)}$

$= -\frac{5}{4}$

c) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3 - 7}{-5 - (-5)}$

$= -\frac{4}{0}$

So the gradient is undefined.

d) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-5 - (-9)}{4 - 0}$

$= \frac{4}{4}$

$= 1$

2. **a)** We have the gradient and the y-intercept.

$y = mx + c$

$m = 1$ and $c = 7$

$\therefore y = x + 7$

b) We have the gradient and one point.

$y = mx + c$

$m = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}x + c$

Substitute in the point (4; 0).

$0 = -\frac{1}{2}(4) + c$

$\therefore c = 2$

$\therefore y = -\frac{1}{2}x + 2$

c) We have the y-intercept and one point.

$y = mx + \mathrm{c}$

$c = 8$

$\therefore y = mx + 8$

Substitute in the point (1; −2):

$-2 = m(1) + 8$

$\therefore m = -10$

$\therefore y = -10x + 8$

d) We have the y-intercept and one point.

$y = mx + c$

$c = 0$

$\therefore y = mx$

Substitute in the point (−1; −6).

$-6 = m(-1)$

$\therefore m = 6$

$\therefore y = 6x$

e) We have two points.

$y = mx + c$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-1 - 5}{-10 - 2}$

$= \frac{6}{12}$

$= \frac{1}{2}$

$\therefore y = \frac{1}{2}x + c$

Substitute in the point (2; 5).

$5 = \frac{1}{2}(2) + c$

$\therefore c = 4$

$\therefore y = \frac{1}{2}x + 4$

f) We have two points.

$y = mx + c$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{4 - (-2)}{7 - (-4)}$

$= \frac{6}{11}$

$\therefore y = \frac{6}{11}x + c$

Substitute in the point (7; 4).

$4 = \frac{6}{11}(7) + c$

$\therefore c = \frac{2}{11}$

$\therefore y = \frac{6}{11}x + \frac{2}{11}$

g) $2y + x = 7$

$\therefore y = -\frac{1}{2}x + \frac{7}{2}$

$\therefore m = -\frac{1}{2}$

We have the gradient and one point.

$y = mx + c$

$m = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}x + c$

Substitute in the point (−1; 1):

$1 = -\frac{1}{2}(-1) + c$

$\therefore c = \frac{1}{2}$

$\therefore y = -\frac{1}{2}x + \frac{1}{2}$

h) $5y = 4x$

$\therefore y = \frac{4}{5}x$

$\therefore m = -\frac{5}{4}$

We have the gradient and the y-intercept.

$y = mx + c$

$m = -\frac{5}{4}$ and $c = -12$

$\therefore y = -\frac{5}{4}x - 12$

3. A: By inspection, $y = -\frac{4}{3}x + 4$

B: By inspection, $x = 3$

C: C ∥ A, so $m = -\frac{4}{3}$

$c = 0$

$\therefore y = -\frac{4}{3}x$

D: By inspection, $y = \frac{2}{5}x + 2$

E: E ⊥ D, so $m = -\frac{5}{2}$

$c = -4\frac{1}{2}$

$\therefore y = -\frac{5}{2}x - 4\frac{1}{2}$

TOPIC 2 Indices and logarithms

Introduction

In this topic, students will revise the laws of indices, prove the laws of logarithms, prove the change of base law, revise how to use the logarithm and antilogarithm tables and use logarithm laws to solve equations that contain logarithms. Students have been working with indices for years and now they will work with logarithms and indices together to emphasise the relationship between them.

Preparation

Prepare charts to display in the classroom:
- the laws of indices and examples
- the logarithm laws and examples
- the logarithm and antilogarithm tables.

Common difficulties

Students often confuse adding numbers to show multiplication and multiplying numbers when working with powers. For example, $3a = a + a + a$ and $a^3 = a \times a \times a$.

Students are often tempted to multiply bases when multiplying powers. For example, $2^3 \times 2^4 = 2^{3+4}$ or 2^7, and $2^3 \times 2^4 \neq 4^7$ or 4^{12} or 2^{12}!

It is important that students understand this section of the work properly and how the laws work. It will help students consolidate the theory if they have to derive the laws and give many simple examples.

Students need to be confident when changing exponents into logarithm form and numbers in logarithm form into exponents. For example, if $a = b^c$, then $\log_b c = a$. The log is the index!

When students use a common factor or trinomial factorising when finding factors, they need to check the brackets in the product to ensure that they factorised correctly. They need to check the signs in the brackets to ensure that an answer is correct.

When factorising a trinomial such as $4^x - 9 \times 2^x + 8$, students often use the rules incorrectly:

$(2^x - 1)(2^x - 8) = (2^x)^2 - 8 \times 2^x - 1 \times 2^x + 8$. Some students think that 8×2^x equals to 16^x but that shows a lack of basic understanding of the index laws.

Introducing students to the topic

This is one section of work where many students struggle when working through the basic index laws. It is thus important to use many simple examples. As long as the students stick to the basic laws, they will then be able to handle the difficult questions. Explain that logarithms are exponents. Ask students to write down the index laws and the log laws and note how they are connected. Work through the worked examples with the class and ask the students to complete the first exercise.

Revise the definition of a logarithm with the class and give examples that show how to write equations in logarithm form or in index form. Revise the logarithm laws and show the students how to prove each log law using the definition of a logarithm to change log form into index form. Emphasise that all these laws only apply to logarithms that have the same base.

Revise the change of base law and work through the proof with the class. Encourage students to change the base of logarithms to base 10 so that they can use logarithm tables to solve problems. Use examples of numbers that are larger than 1 and smaller than 1 to show how to read logarithm tables. Revise the use of the differences columns for four-digit numbers. Work through the worked examples that include multiplication and division as well as the rules for calculating powers and roots.

Lastly, use the laws of logarithms and also basic algebra knowledge to solve equations that contain logarithms.

Work through the worked examples and encourage students to look at each equation and decide which laws to use and whether it is necessary to factorise an equation in order solve it.

Answers

Exercise 2.1 (page 21)

1. **a)** 4^7 **b)** 3^4 **c)** 3^6
d) $2^{-10} = \frac{1}{2^{10}}$ **e)** 3^3 **f)** $3^{\frac{5}{6}}$

2. **a)** $\frac{9}{16}$ or $\frac{3^2}{2^4}$ **b)** $\frac{8}{9}$ or $\frac{2^3}{3^2}$ **c)** $\frac{1}{4}$
d) $\frac{5}{2}$ **e)** 27 or 3^3 **f)** $\frac{1}{5}$

3. **a)** $\frac{1}{3a^3}$ **b)** $4a^2$ **c)** $\frac{12}{a^3}$
d) $\frac{9}{5}$ **e)** 3 **f)** 1

Exercise 2.2 (page 22)

1. **a)** 25 **b)** −64 **c)** $\frac{1}{6}$
d) 9 **e)** $\frac{1}{125}$ **f)** 243

2. **a)** −2 **b)** $\frac{3}{5}$ **c)** $\frac{-3}{2}$
d) 3 **e)** $\frac{5}{4}$ **f)** 9

3. **a)** 2 **b)** $\frac{3}{2}$; −1 **c)** −2
d) $2^x = 8$ or $2^x = 1$, so, $x = 3$ or $x = 0$
e) $x = \frac{1}{3}$ **f)** $x = -\frac{1}{2}$

Exercise 2.3 (page 25)

1. **a)** $\log_3 81 = 4$ **b)** $\log_8 2 = \frac{1}{3}$ **c)** $\log 1\,000 = 3$
d) $\log_{25} 1 = 0$ **e)** $\log_8 32 = x$ **f)** $\log_a c = \mathrm{b}$

2. **a)** $5^3 = 125$ **b)** $2^{-2} = \frac{1}{4}$ **c)** $10^4 = 10\,000$
d) $3^0 = 1$ **e)** $4^x = 100$ **f)** $\mathrm{b}^c = a^2$

3. **a)** 4 **b)** $\frac{3}{2}$ **c)** −4
d) −1 **e)** 1 **f)** 3

4. **a)** log 15 **b)** log 2 **c)** log 18
d) log 72 **e)** $\log \frac{4}{3}$ **f)** log 50

5. **a)** 1.6902 **b)** 1.4472 **c)** 0.30105

6. **a)** $\frac{3}{2}$ **b)** 0 **c)** $\frac{2}{3}$
d) −2 **e)** 2 **f)** 1

Exercise 2.4 (page 26)

1. **a)** $\frac{1}{3}$ **b)** $-\frac{1}{3}$ **c)** 5 **d)** $\frac{7}{4}$

2. $\frac{\log 15}{\log 3}$

3. 1

4. **a)** Right side $= \frac{\log 4}{\log 3} \div \frac{\log 4}{\log 2}$

$= \frac{\log 4}{\log 3} \times \frac{\log 2}{\log 4}$

$= \frac{\log 2}{\log 3}$

$= \log_3 2$

The right side equals the left side.

b) Right side $= 1 \div \frac{\log a}{\log b}$

$= 1 \times \frac{\log b}{\log a}$

$= \log_a b$

The right side equals the left side.

c) Right side $= -\frac{\log b}{\log a}$

$= \frac{\log b}{\log a^{-1}}$

$= \frac{\log b}{\log \frac{1}{a}}$

$= \log_{\frac{1}{a}} b$

The right side equals the left side.

5. 4

Exercise 2.5 (page 29)

The answers below were calculated using the log tables. Answers calculated using a calculator differ slightly.

1. **a)** 448.3 **b)** 1.241 **c)** 3.448
d) 85.33 **e)** 3.225 **f)** 11.77

2. **a)** 10.34 **b)** 0.4782 **c)** 4.234
d) 2.767 **e)** 2.014 **f)** 4.553

3. **a)** 3.146 **b)** 452.7 **c)** 2.465
d) 3.943 **e)** 8.874 **f)** 3.713

Exercise 2.6 (page 30)

1. **a)** 64 **b)** 0 **c)** $\frac{1}{100}$ **d)** -2

2. **a)** 0 **b)** 2.98 **c)** −7.78 **d)** 7.15

3. **a)** 100 **b)** 18 **c)** 8 **d)** 3

4. **a)** −3.11 **b)** 3.62; 1.38
c) 4, $x \neq 0$ **d)** 3, 9

Assess your progress (page 30)

1. **a)** $3^3 = 27$ **b)** $5^{-2} = \frac{1}{25}$ **c)** 4 **d)** 1

2. **a)** 1.621 **b)** 0; 8 **c)** 0.442 **d)** $\frac{2}{7} = 0.2857$

3. **a)** $\frac{x}{y} = \frac{3}{2}$
b) $x = -2$ and $y = 2$; $x + y = 0$

4. 2

5. **a)** 1.292 **b)** $\frac{1}{2}$

6. Right side $= \log_5 15$

$$= \frac{\log 15}{\log 5}$$
$$= \frac{1.176}{0.6990}$$
$$= 1.682$$

7.
$$\frac{\log x}{\log 3} = \log x + 1$$
$$\therefore \frac{\log x}{\log 3} - \log x = 1$$
$$\therefore \frac{\log x - \log 3 \log x}{\log 3} = 1$$
$$\therefore \frac{\log x(1 - \log 3)}{\log 3} = 1$$
$$\therefore \log x = \frac{\log 3}{1 - \log 3}$$

8.
$$x^{\frac{1}{2}} = 10^{\bar{1}.9274}$$
$$\therefore x = 10^{\bar{1}.8548}$$
$$\therefore x = 0.7158$$

9. 0.588

10. $\log_{15} 9 + \log_{15} 25 = \log_{15} 225 = 2$

11. **a)** $\log_x \frac{1}{2} = -1$; $x = 2$

b) $\log(x + 1) = \log x + \log 10$

$\therefore \log(x + 1) = \log 10x$

$\therefore x + 1 = 10x$

$\therefore x = \frac{1}{9}$

c) $x^2 - 7x + 6 = -6$

$x^2 - 7x + 12 = 0$

$(x - 4)(x - 3) = 0$

$x = 4$ or $x = 3$

12. a) $\log 12^{x - 1} = \log 4^x$

$\therefore (x - 1)\log 12 = x \log 4$

$\therefore x \log 12 - \log 12 - x \log 4 = 0$

$\therefore x(\log 12 - \log 4) = \log 12$

$\therefore x \log 3 = \log 12$

$\therefore x = \frac{\log 12}{\log 3}$

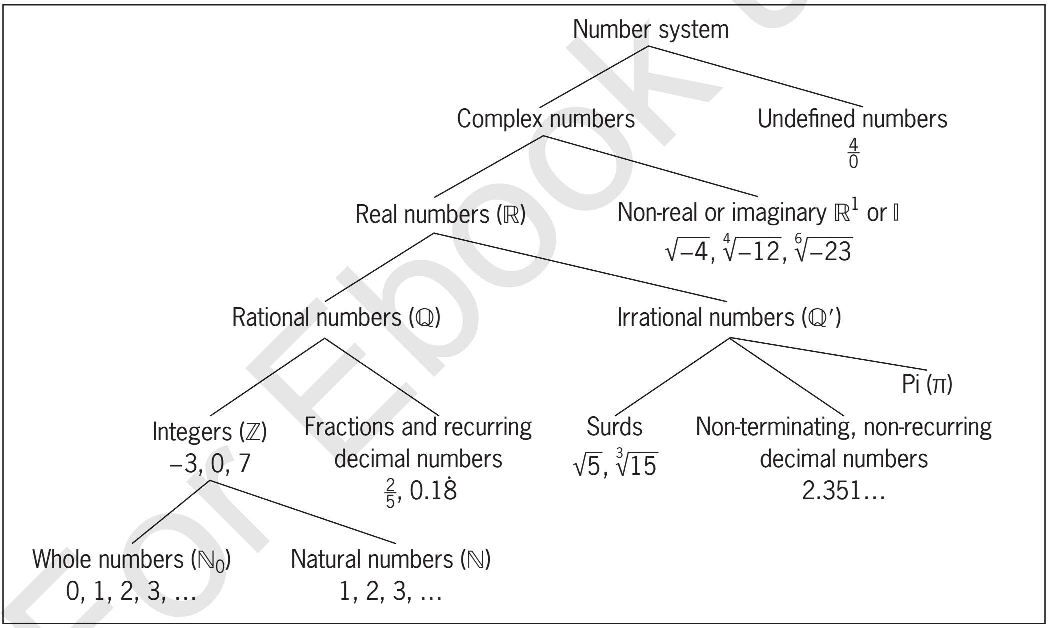
b) 2.26

TOPIC 3 Surds

Introduction

In this topic, students will differentiate between rational and irrational numbers, learn definitions for surds, revise surd rules, simplify numerical surds, add and subtract surds, multiply and divide surds, find the conjugates of binomial surds and rationalise denominators.

Explain to students how rational and irrational numbers fit into the family tree of numbers. You could make a poster of the whole classification of numbers in the number system. Part of the diagram is in the Student's Book on page 33.



Preparation

Prepare posters to display in the classroom:
- the family tree of numbers
- rules for adding and subtracting surds
- rules for multiplying and dividing surds
- conjugates with monomial and binomial denominators.

Common difficulties

Often students do not understand the classification of the different types of real numbers. They need to realise that real numbers are made up of different families of numbers from real numbers to natural numbers.

Students who prefer working with indices to working with surds, can often change the surds to index form and then use the index laws when solving problems. If students practise changing numbers from surd form to index form, they can use the index rules, which are often easier to use than the surd laws.

Students need to remember the rules for multiplying and dividing surds and remember that these rules do not work for addition and subtraction.

When rationalising a denominator, students must realise that multiplying by 1 in any form does not change the value of a fraction.

Introducing students to the topic

Emphasise that we can write recurring decimal numbers and terminating decimal numbers as fractions, but we cannot write non-terminating decimal numbers (such as π) in fraction form. When we can write the root of a number as a rational number, we can write the irrational number in surd form.

It is important to show the students that a root of a negative number can be negative if the root is odd. An even root of a negative number gives a non-real answer. This work is beyond the scope of this curriculum. For example, $\sqrt{-6}$ and $\sqrt[4]{-12}$ have non-real or imaginary answers.

When comparing surds, students may find it easier to write the surds in index form and compare the powers. For example, when comparing $\sqrt[5]{6}$ and $\sqrt{3}$:

$\sqrt[5]{6} = 6^{\frac{1}{5}} = 6^{\frac{2}{10}} = (6^2)^{\frac{1}{10}} = 36^{\frac{1}{10}}$ and

$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{5}{10}} = (3^5)^{\frac{1}{10}} = 729^{\frac{1}{10}}$. So, $\sqrt{3} > \sqrt[5]{6}$.

It is important that students realise that they can use different methods to solve problems. They need to use the method that they understand and that works better for them.

Work through the worked examples and ask students to answer the questions in the exercises.

Answers

Exercise 3.1 (page 35)

1. **a)** $0.365, \frac{\sqrt{3}}{\sqrt{12}}, (\sqrt{7})^2, \sqrt[3]{-27}, -\sqrt{25}, \frac{3}{8}, 3.14, -43, 6^{-1}$

b) $\sqrt{5}, \sqrt[3]{12}, \pi, -\sqrt{45}, \sqrt{-50}, \sqrt{\frac{14}{9}}, \sqrt{5} + 1$

2. **a)** False **b)** True **c)** True
d) False **e)** False **f)** True

3. **a)** $2 < \sqrt{8} < 3$ **b)** $4 < \sqrt{17} < 5$
c) $-2 < -\sqrt{3} < -1$ **d)** $-8 < -\sqrt{55} < -7$
e) $4 < \sqrt[3]{85} < 5$ **f)** $-4 < \sqrt[3]{-38} < -3$

4. **a)** $\sqrt[3]{3}$ **b)** $\sqrt{3}$ **c)** $\sqrt[3]{4}$ **d)** $5\sqrt{2}$

5. **a)** $\sqrt{3}, \sqrt[3]{8}, 2.5$ **b)** $3, \pi, \sqrt{12}$
c) $\sqrt{33}, 5\frac{3}{4}, 5.8$ **d)** $-\sqrt{5}, -2.2, 0$

Exercise 3.2 (page 37)

1. **a)** $\sqrt{3}\sqrt{5}$ **b)** $\sqrt{7}\sqrt{11}$
c) $-\sqrt{9}\sqrt{4}\sqrt{2} = -6\sqrt{2}$ **d)** $\sqrt[3]{27}\sqrt[3]{2} = 3\sqrt[3]{2}$
e) $\sqrt[3]{-8}\sqrt[3]{3} = -2\sqrt[3]{3}$ **f)** $\sqrt[5]{32}\sqrt[5]{2} = 2\sqrt[5]{2}$

2. **a)** $\sqrt{45}$ **b)** $\sqrt{150}$ **c)** $\sqrt{48}$ **d)** $\sqrt{32}$

3. **a)** $\sqrt{11}$ **b)** $\sqrt[3]{3}$ **c)** $1 + \sqrt{2}$ **d)** $\sqrt{2} + 2$

Exercise 3.3 (page 37)

1. **a)** $\sqrt{15}$ **b)** $12\sqrt{12} = 24\sqrt{3}$
c) $\sqrt{3}$ **d)** $3\sqrt{3}$
e) $7\sqrt{35}$ **f)** $3\sqrt{3}$

2. **a)** 40 **b)** 27 **c)** 8 **d)** 30

3. **a)** 4 **b)** 3 **c)** 4 **d)** 21

Exercise 3.4 (page 38)

1. **a)** $10\sqrt{3}$ **b)** $4\sqrt{2}$ **c)** $5\sqrt{5}$
d) $4\sqrt{7}$ **e)** $6\sqrt{15}$ **f)** $7\sqrt{13}$

2. a) $4\sqrt{11}$ b) $-2\sqrt{2}$ c) $5\sqrt{3} + 5\sqrt{2}$
d) $3\sqrt{3} - 3$ e) $3\sqrt[3]{2}$ f) $2\sqrt[3]{3}$

3. a) $3\sqrt{2} + 2\sqrt{2} - 5\sqrt{2} = 0$
b) $4\sqrt{5} - 2\sqrt{5} - 3\sqrt{5} = -\sqrt{5}$
c) $6\sqrt{3} - 2\sqrt[3]{3} - 3\sqrt{3} = 3\sqrt{3} - 2\sqrt[3]{3}$
d) $11\sqrt{2} + 10\sqrt{2} - 12\sqrt{2} = 9\sqrt{2}$
e) $3\sqrt[3]{3} - 2\sqrt[3]{3} - \sqrt[3]{3} = 0$
f) $2\sqrt[3]{2} + 4\sqrt[3]{2} - 5\sqrt[3]{2} = -\sqrt[3]{2}$

4. a) $\frac{7\sqrt{2} - 5\sqrt{2}}{3\sqrt{2}} = \frac{2}{3}$ b) $\frac{5\sqrt{5} - 3\sqrt{5} - 2\sqrt{5}}{\sqrt{5}} = 0$
c) $\frac{8\sqrt{2} - 5\sqrt{2}}{4\sqrt{2} + 2\sqrt{2}} = \frac{1}{2}$ d) $\frac{3\sqrt{11} - \sqrt{11}}{2\sqrt{11}} = 1$

5. 70

Exercise 3.5 (page 39)

1. a) $\sqrt{5} + \sqrt{3}$ b) $\sqrt{2} - \sqrt{6}$ c) $\sqrt{7} - 1$
d) $4 + \sqrt{11}$ e) $4\sqrt{5} + \sqrt{5}$ f) $\sqrt{6} - 3\sqrt{2}$

2. a) 3 b) 23 c) −19
d) 4 e) 43 f) −39

Exercise 3.6 (page 40)

1. a) $\frac{5\sqrt{3}}{3}$ b) $\frac{-2\sqrt{5}}{5}$ c) $\frac{2\sqrt{35}}{7}$
d) $\frac{11\sqrt{105}}{15}$ e) $\frac{3\sqrt{2}}{4}$ f) 1

2. a) $\frac{3 - \sqrt{5}}{2}$ b) $\frac{-\sqrt{7} - 1}{6}$
c) $-8 + 7\sqrt{2}$ d) $\frac{2\sqrt{3} - 3\sqrt{2}}{-2}$
e) $3\sqrt{3} + 3\sqrt{2} - \sqrt{6} - 2$ f) $\frac{8\sqrt{2} + 4}{7}$

3. −16

4. $\frac{(7 + 4\sqrt{3}) - (7 - 4\sqrt{3})}{(7 - 4\sqrt{3})(7 + 4\sqrt{3})} = 8\sqrt{3}$

5. $\frac{1}{1 + \sqrt{2} - \sqrt{3}} = \frac{\sqrt{2} + 2 + \sqrt{6}}{4}$
Hint: First multiply the fraction by $\frac{1 + \sqrt{2} + \sqrt{3}}{1 + \sqrt{2} + \sqrt{3}}$ and then multiply the answer by $\frac{2\sqrt{2}}{2\sqrt{2}}$.

Assess your progress (page 41)

1. **a)** Irrational **b)** Irrational **c)** Rational
   d) Irrational **e)** Rational **f)** Rational

2. **a)** $\sqrt{3}$ **b)** $3\sqrt{2}$
   c) $2\sqrt[3]{2}$ **d)** $0 + \sqrt{4} - 2 = 0$

3. **a)** 1 **b)** 89
   c) -1 **d)** 53

4. **a)** $\frac{2\sqrt{21}}{7}$ **b)** 2
   c) 3 **d)** 2
   e) $\sqrt{2} - 1$ **f)** $3\sqrt{2} + 2\sqrt{3}$

5. **a)** 64 **b)** 7
   c) 14 **d)** $\frac{3(5 + \sqrt{2})}{23}$

6. -1

7. 1

8. 1

TOPIC 4

Trigonometry

Introduction

In this topic, students will revise Pythagoras' theorem, apply surd rules to problems that involve the trigonometric ratios of 0°, 30°, 45°, 60° and 90°, draw graphs of sine and cosine angles, interpret trigonometric graphs and solve trigonometric equations.

Students have used the sine and cosine rules to solve trigonometric problems and now they will work with special triangles and use their knowledge of surds to calculate the trigonometrical ratios sine, cosine and tangent for angles 0°, 30°, 45°, 60° and 90°. They will also work with the graphs for the functions $y = \sin x$ and $y = \cos x$.

Students will need to be able to plot points accurately so that they can draw the trigonometric functions.

Students will solve simultaneous linear and trigonometric equations graphically.

Refer your students to the trigonometric tables at the back of the Student's Book. Students worked with these tables in SS1 and SS2. Check how well students remember how to use the different tables.

Common difficulties

Remind students to insert the unit with the final answer when they solve problems. If a problem does not include units of length, the students should write only the answer, or the number followed by the word "units".

Remind your students to ensure that their calculators are always in Degree mode for trigonometric calculations.

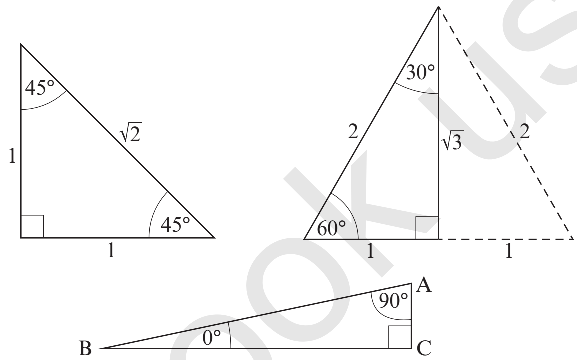
Students do not always know the trigonometric ratio definitions and they then give incorrect ratios. Suggest that they use a memory aid such as SOHCAHTOA to help them remember the definitions of the trigonometric ratios.

Preparation

Students will need graph boards, graph books, pencils, rulers, sticks and twine (for creating right-angled triangles of different lengths)

Prepare and display the following posters in the classroom:

- Prepare a poster that shows a right-angled triangle and the trigonometric definitions for sine, cosine and tangent (you can use the triangle on page 48 in the Student's Book).
- Prepare a poster to show triangles for the special angles 0°, 30°, 45° and 60° and 90°.



- Make a copy of the table on page 48 in the Student's Book and encourage students to learn the ratios.
- Prepare a poster of the graphs of the trigonometric functions, $y = \sin x$ and $y = \cos x$ for $0° \le x \le 360°$.

Introducing students to the topic

Revise Pythagoras' theorem with particular reference to using surds when you have to find the size of a missing side in a triangle.

Briefly revise the trigonometry students worked with in SS1. Make sure that they remember and understand the definitions of the three basic trigonometric ratios (sine = $\frac{\text{opposite}}{\text{hypotenuse}}$, cosine $\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and tangent $\theta = \frac{\text{opposite}}{\text{adjacent}}$). Work through the worked examples with the class and ask students to complete Exercise 4.1

Answers

Exercise 4.1 (page 44)

1. NM = 14

2. **a)** 10 **b)** 25
   c) $\sqrt{394}$ **d)** $2x^2 = 100$; so $x = 5\sqrt{2}$

3. **a)** $b = 8$ **b)** $z = 3\sqrt{3}$
   c) $y = 5$ **d)** $r = 36$

4. **a)** △FGH is not right-angled.
   b) △IJK is right-angled.

5. $z = 5$

6. **a)** $\sqrt{89}$ **b)** $3\sqrt{34}$ **c)** $2\sqrt{58}$

7. **a)** $2\sqrt{11}$ **b)** $6\sqrt{7}$ **c)** $10\sqrt{5}$

8. $y = \sqrt{2}$

Exercise 4.2 (page 49)

1. **a)** 0 **b)** 1 **c)** 0
   d) $\frac{1}{2}$ **e)** $\frac{\sqrt{3}}{2}$ **f)** $\frac{1}{\sqrt{3}}$
   g) $\frac{1}{\sqrt{2}}$ **h)** $\frac{1}{\sqrt{2}}$ **i)** 1
   j) $\frac{\sqrt{3}}{2}$ **k)** $\frac{1}{2}$ **l)** $\frac{\sqrt{3}}{1}$
   m) 1 **n)** 0 **o)** undefined

2. **a)** The ratios increase from 0 to 1.
   b) The ratios decrease from 1 to 0.
   c) The ratios increase from 0 to infinity.
   d) tan 90° is undefined.

3. **a)** $x = 8$ **b)** $x = \sqrt{89}$

4. **a)** $\sin\theta = \frac{8}{17}$; $\cos\theta = \frac{15}{17}$; $\tan\theta = \frac{8}{15}$
   b) $\sin\theta = \frac{5}{\sqrt{89}}$; $\cos\theta = \frac{8}{\sqrt{89}}$; $\tan\theta = \frac{5}{8}$

5. **a)** 1 **b)** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **c)** $\frac{2}{3}$ **d)** 1
   e) 1 **f)** 1 **g)** 3 **h)** $\frac{1}{4}$
   i) 1 **j)** $\frac{1}{3}$

Exercise 4.3 (page 52)

1. **a)** (90°; 1) **b)** (180°; −1)

2. **a)** 1 **b)** 360° **c)** (−1; 1]

3. **a)** 0.5 **b)** −1 **c)** −1 **d)** 1

4. **a)** 0°, 180°, 360° **b)** 60°, 300°
c) 270° **d)** 120°, 240°

5. **a)** $\theta = 45°; 135°$ **b)** $\theta = 0°; 360°$
c) $\theta = 30°; 150°$ **d)** $\theta = 120°; 240°$

6. **a)** $\theta = 285°$ **b)** $\theta = 160°$
c) $\theta = 324°$ **d)** $\theta = 226°$

7. **a)** 36.9° and 143.1° **b)** 66.4° and 293.6°
c) 210° and 330° **d)** 143.1° and 216.9°

Assess your progress (page 54)

1. **a)** AC = 12 **b)** AC = $2\sqrt{241}$

2. DE = $\sqrt{1\,113}$

3. BC = $6\sqrt{3}$

4. **a)** $x = 5$ **b)** $y = 12$ **c)** $z = 7$

5. $x = 2$

6. $r = 5$ cm

7. **a)** $x = 8$
b) $\sin\theta = \frac{8}{17}$; $\cos\theta = \frac{15}{17}$; $\tan\theta = \frac{8}{15}$

8. **a)** $\sqrt{2}$ **b)** $\frac{3\sqrt{3}}{2}$ **c)** 1 **d)** 2
e) 0 **f)** 2 **g)** 1 **h)** 1

9. **a)** (0°; 1); (360°; 1) **b)** (270°; −1)

10. **a)** 1 **b)** 360° **c)** (−1; 1]

11. **a)** $x = 45°; 315°$ **b)** $x = 150°; 210°$
c) $x = 210°; 330°$ **d)** $x = 45°; 225°$

12. **a)** $\theta = 345°$ **b)** $\theta = 130°$
c) $\theta = 220°$ **d)** $\theta = 235°$

TOPIC 5 Matrices and determinants

Introduction

In this topic, students will define matrices, state the order of a matrix, learn notation of matrices, define types of matrices, add and subtract matrices, multiply matrices by scalar quantities, multiply matrices with matrices, find the transpose of a matrix, calculate the determinant of a square matrix and apply matrix methods to solve simultaneous equations.

A matrix is a rectangular array of numbers, symbols or expressions with numbers arranged in rows and columns. The items in a matrix are called its elements or entries.

In this topic, students work with rectangular, row, column, square, null or zero, diagonal, identity or unit and equal matrices.

We can add and subtract matrices of the same order and we can multiply matrices by a scalar quantity, which can be a positive or negative rational number. Matrix multiplication is not commutative. Students will learn about the transposes and determinants of matrices. The inverse of a matrix is the reciprocal of the matrix. If we multiply any matrix by its inverse, the result is the identity matrix I. The inverse of a matrix A is A^{-1}.

Students can use the determinant or simultaneous equations to calculate the inverse of a matrix and they can use matrices to solve simultaneous equations with two unknowns.

Common difficulties

Students often confuse the rows and the columns of matrices and inverses and determinants. Encourage the students to list all the new vocabulary for this section of work and learn these terms. Work through the examples carefully with the class before asking them to tackle an exercise.

Take care with double negatives; for example, $-1 + (-4) = -5$.

Preparation

Prepare posters of matrix charts that illustrate:

- the different types of matrices
- the rules for addition and subtraction
- scalar multiplication
- multiplying 2 by 2 matrices
- multiplying 3 by 3 matrices
- transposes, determinants and inverses
- information from the internet about matrices.

Introducing students to the topic

Revise basic mathematical operations with students − adding and subtracting with negative numbers and fractions, and multiplying negative numbers and fractions. Work through a few examples using information that is relevant to students (such as test results or students' heights). Explain matrices and the different types of matrices.

Introduce each section by working through the worked examples before asking students to complete the exercises.

Answers

Exercise 5.1 (page 59)

1. **a)** 4×4 **b)** 2×3 **c)** 2×2
   d) 2×6 **e)** 1×4 **f)** 4×2

2. **a)** E **b)** F
   c) A, B, C and I **d)** B
   e) D, E and F **f)** C and I
   g) I

3. $a = 11$, $b = 14$, $c = 6$, $d = 5$

4. $\begin{pmatrix} 10 & 8 & 12 & 6 \end{pmatrix}$

5. $\begin{pmatrix} 39 & 58 \\ 43 & 52 \\ 62 & 59 \\ 55 & 67 \end{pmatrix}$

6. **a)** $\begin{pmatrix} 3 & 2 & 1 & 0 & 9 & 4 \\ 3 & 1 & 2 & 0 & 7 & 1 \\ 3 & 1 & 0 & 2 & 4 & 6 \\ 3 & 0 & 1 & 2 & 3 & 8 \end{pmatrix}$ **b)** 4×6

c) Row 3 column 5 **d)** A

Exercise 5.2 (page 61)

1. **a)** A: 2×2, B: 4×2, C: 2×2, D: 2×3

b) A and C

c) $\begin{pmatrix} 5 & 20 \\ 12.5 & 19 \end{pmatrix}$

2. **a)** $\begin{pmatrix} 26 & 12 \\ 2 & 8 \end{pmatrix}$ **b)** $\begin{pmatrix} 10 & 8 \\ 5 & -4 \end{pmatrix}$

c) $\begin{pmatrix} -10 & -8 \\ -5 & 4 \end{pmatrix}$ **d)** $\begin{pmatrix} 6 & -4 \\ -8 & 16 \end{pmatrix}$

e) $\begin{pmatrix} -16 & -4 \\ 3 & -12 \end{pmatrix}$ **f)** $\begin{pmatrix} -6 & 4 \\ 8 & -16 \end{pmatrix}$

3. $\begin{pmatrix} 14 & 39 \\ 30 & 14 \\ 35 & 20 \\ 33 & 25 \end{pmatrix}$

4. $a = 2$, $b = -2$, $c = -1$

5. **a)** Day 1: $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, Day 2: $\begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

b) 13 **c)** $\begin{pmatrix} 5 & 5 & 3 \\ 5 & 5 & 3 \\ 3 & 3 & 7 \end{pmatrix}$

6. **a)** $\begin{pmatrix} 26 & 32 & 18 \\ 65 & 72 & 40 \\ 40 & 48 & 26 \\ 18 & 31 & 12 \\ 36 & 42 & 20 \end{pmatrix}$ **b)** $\begin{pmatrix} 21 & 24 & 14 \\ 29 & 23 & 12 \\ 19 & 20 & 12 \\ 14 & 25 & 9 \\ 17 & 23 & 15 \end{pmatrix}$

c) $\begin{pmatrix} 5 & 8 & 4 \\ 36 & 49 & 28 \\ 21 & 28 & 14 \\ 4 & 6 & 3 \\ 19 & 19 & 5 \end{pmatrix}$ **d)** Shop B

Exercise 5.3 (page 63)

1. **a)** $\begin{pmatrix} 4 & 8 \\ 12 & 16 \end{pmatrix}$ **b)** $\begin{pmatrix} -2 & 12 \\ 6 & -6 \end{pmatrix}$ **c)** $\begin{pmatrix} 3.6 & 6.6 \\ 12 & -3.3 \end{pmatrix}$

d) $\begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix}$ **e)** $\begin{pmatrix} 20 & 16 \\ 8 & 12 \end{pmatrix}$ **f)** $\begin{pmatrix} 20 & 16 \\ 8 & 12 \end{pmatrix}$

2. **a)** $\begin{pmatrix} 25 & 10 \\ -10 & 5 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & -9 \\ 6 & -3 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 6 \\ 12 & 2.5 \end{pmatrix}$

d) $\begin{pmatrix} 4 & 8 \\ 6 & -2 \end{pmatrix}$ **e)** $\begin{pmatrix} 12 & 18 \\ -6 & 24 \end{pmatrix}$ **f)** $\begin{pmatrix} 4 & -8 \\ 2 & 6 \end{pmatrix}$

3. **a)** $\begin{pmatrix} 4 & 12 \\ 8 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 15 \\ 12 & 6 \end{pmatrix}$ **c)** $\begin{pmatrix} 6 & 9 \\ 7 & 8 \end{pmatrix}$

d) $\begin{pmatrix} 12 & 44 \\ 32 & 12 \end{pmatrix}$ **e)** $\begin{pmatrix} 26 & 46 \\ 36 & 36 \end{pmatrix}$ **f)** $\begin{pmatrix} 42 & 72 \\ 54 & 51 \end{pmatrix}$

Exercise 5.4 (page 65)

1. **a)** $\begin{pmatrix} 11 \\ 21 \end{pmatrix}$

b) Impossible; you cannot multiply a 2 × 1 matrix by a 2 × 2 matrix.

2. **a)** $\begin{pmatrix} 42 & 46 \\ 28 & 24 \end{pmatrix}$ **b)** $\begin{pmatrix} 58 & 2 \\ 22 & -13 \end{pmatrix}$ **c)** $\begin{pmatrix} -12 & -45 \\ -21 & -57 \end{pmatrix}$

d) $\begin{pmatrix} -20 & -10 \\ 30 & 18 \end{pmatrix}$ **e)** $\begin{pmatrix} 5 & -\frac{15}{4} \\ -4 & 3 \end{pmatrix}$ **f)** $\begin{pmatrix} -49 & -38 \\ 10 & \frac{35}{2} \end{pmatrix}$

3. **a)** $\begin{pmatrix} 20 & 2 \\ 14 & -20 \end{pmatrix}$ **b)** $\begin{pmatrix} 9 & -5 \\ -5 & -9 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 10 \\ -9 & 14 \end{pmatrix}$

d) $\begin{pmatrix} 6 & -7 \\ 2 & 15 \end{pmatrix}$ **e)** $\begin{pmatrix} -10 & 26 \\ -5 & -2 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$

4. **a)** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ **c)** $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

d) $\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ **e)** $\begin{pmatrix} 15 & 22 \\ 33 & 48 \end{pmatrix}$ **f)** $\begin{pmatrix} 16 & 24 \\ 36 & 52 \end{pmatrix}$

g) $\begin{pmatrix} 14 & 20 \\ 30 & 44 \end{pmatrix}$ h) $\begin{pmatrix} 14 & 20 \\ 30 & 44 \end{pmatrix}$ i) $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

j) $\begin{pmatrix} 35 & 50 \\ 75 & 110 \end{pmatrix}$

5. **a)** $\begin{pmatrix} 12 & -3 & -7 \\ -5 & -1 & 4 \\ 1 & 13 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} 10 & -12 & 2 \\ -5 & 4 & 2 \\ 11 & -3 & -1 \end{pmatrix}$

Exercise 5.5 (page 70)

1. **a)** $\mathbf{A}^T = (12 \quad 13 \quad 52 \quad 37)$ **b)** $\mathbf{B}^T = \begin{pmatrix} 3 & 25 \\ 12 & 8 \\ -9 & 11 \\ 14 & -6 \end{pmatrix}$

c) $\mathbf{C}^T = \begin{pmatrix} 19 & 32 \\ 42 & -31 \end{pmatrix}$ **d)** $\mathbf{D}^T = \begin{pmatrix} 0 & 39 & 10 \\ -31 & 53 & 18 \\ 22 & -42 & 13 \end{pmatrix}$

2. **a)** 3 **b)** 5 **c)** −16 **d)** 17

3. **a)** 11 **b)** 4 **c)** 44
d) $\begin{pmatrix} 8 & -12 \\ 7 & -5 \end{pmatrix}$ **e)** 44 **f)** 44

4. **a)** 5 **b)** −306 **c)** 31 **d)** −28

5. **a)** 9 **b)** −3
c) −27 **d)** $\begin{pmatrix} -7 & 3 & 1 \\ 0 & -3 & 0 \\ -2 & 6 & -1 \end{pmatrix}$
e) −27 **f)** −27

6. **a)** −76 **b)** −76

Exercise 5.6 (page 73)

1. **a)** $a = 2, b = 7, c = -3, d = -6$
b) $a = \frac{1}{2}, b = 3, c = -\frac{3}{2}, d = -3$
c) $a = -1, b = 1, c = 2$
d) $a = 12, b = -10, c = 2, d = 5$

2. $a = 0$ and $b = -2$

3. **a)** $\begin{pmatrix} -3 & 5 \\ 2 & -3 \end{pmatrix}$ **b)** $\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$

c) $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ **d)** Not possible

4. **a)** -2 **b)** $\begin{pmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{pmatrix}$

c) $x = -\frac{3}{2}$ and $y = 6$

5. **a)** $\begin{pmatrix} 1 & \frac{-1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$ **b)** $\begin{pmatrix} \frac{-1}{7} & \frac{-4}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}$

c) $\begin{pmatrix} -3 & -5 \\ -2 & -10 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{4} \\ \frac{1}{10} & \frac{-3}{20} \end{pmatrix}$

d) $\begin{pmatrix} -1 & 4 \\ 1 & -6 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & -2 \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix}$

Assess your progress (page 74)

1. **a)** 4×2 **b)** 2×3 **c)** 4×1 **d)** 3×2

2. **a)** **B** and **F** **b)** **A**, **C**, **D**, **E**, **G**

c) **D** **d)** **E**

e) **B** and **F** **f)** **B**

3. **a)** **A** and **F**, **B** and **D**, **C** and **G**

b) $\begin{pmatrix} 18 & 29 \\ -44 & 32 \\ 51 & -92 \end{pmatrix}$ **c)** $\begin{pmatrix} \frac{1}{6} & \frac{-1}{9} \\ \frac{-1}{12} & \frac{2}{9} \end{pmatrix}$

d) $\left|\begin{pmatrix} 32 & 100 \\ 30 & 60 \end{pmatrix}\right| = -1\,080$ **e)** $\begin{pmatrix} 37 & -54 \\ 106 & -1 \\ -20 & -61 \end{pmatrix}$

f) $\begin{pmatrix} 18 & 20 & -40 \\ -60 & 27 & 19 \end{pmatrix} - 2\begin{pmatrix} 18 & -44 & 51 \\ 29 & 32 & -92 \end{pmatrix}$

$= \begin{pmatrix} -18 & 108 & -142 \\ -118 & -37 & 203 \end{pmatrix}$

g) $2\begin{pmatrix} 2 & 10 \\ 4 & 5 \end{pmatrix} - 3\begin{pmatrix} 8 & 4 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} -20 & 8 \\ -1 & -8 \end{pmatrix}$

h) $\begin{pmatrix} 44 & 70 \\ 28 & 65 \end{pmatrix}$

4. **a)** $a = 8, b = 3, c = 0$ **b)** $d = -2, e = 20, f = 18$

5. **a)** $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

6. **a)** 10 **b)** -1 **c)** -10
d) $\begin{pmatrix} -13 & 10 \\ 1 & 0 \end{pmatrix}$ **e)** -10 **f)** -10

7. **a)** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **b)** $\begin{pmatrix} 40 & -48 \\ -12 & 15 \end{pmatrix}$

8. **a)** $\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 12 - 10 & 8 - 8 \\ -15 + 15 & -10 + 12 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 12 - 10 & -6 + 6 \\ 20 - 20 & -10 + 12 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

c) Square matrices

9. **a)** $\begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix}\begin{pmatrix} -3 & -5 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -15 & -29 \\ -3 & -21 \end{pmatrix}$

b) $\begin{pmatrix} -3 & -5 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} -20 & -46 \\ -2 & -16 \end{pmatrix}$

c) $\begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix}\begin{pmatrix} -1 & 7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 23 \\ 23 & -41 \end{pmatrix}$

d) $\begin{pmatrix} -1 & 7 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 54 \\ 9 & -42 \end{pmatrix}$

e) $\begin{pmatrix} -3 & -5 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} -12 & 9 \\ -6 & 12 \end{pmatrix}$

f) $\begin{pmatrix} -1 & 7 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} -3 & -5 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & -9 \\ -9 & -3 \end{pmatrix}$

10. **a)** 30 **b)** $\begin{pmatrix} -\frac{1}{15} & \frac{1}{5} \\ -\frac{2}{15} & -\frac{1}{10} \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & -6 \\ 4 & -2 \end{pmatrix}$

11. Team D: $8 \times 3 + 2 \times 1 = 26$ points
Team C: $7 \times 3 + 3 \times 1 = 24$ points
Team A: $6 \times 3 + 3 \times 1 = 21$ points
Team B: $5 \times 3 + 4 \times 1 = 19$ points

TOPIC 6

Simultaneous equations

Introduction

In this topic, students will solve simultaneous linear equations, solve simultaneous linear and non-linear equations, solve word problems involving linear and quadratic equations and solve problems on linear equations that involve the capital market.

When revising how to use substitution when solving two linear equations, remind the students that the solution gives the coordinates of the point of intersection of two straight lines. When they have found the first variable, they need to substitute to find the second variable and check that the solutions are correct.

Students work with simultaneous linear and non-linear equations and functions in this topic. Find the point(s) of intersection of a straight line and a parabola or a circle by looking at the graph or by creating simultaneous equations and solving for x and y.

Students then use the method for solving simultaneous equations to solve real-life problems that involve area, ages and numbers. They will also solve travel problems and problems that involve money and the capital market.

Common difficulties

It is easy to make careless mistakes when solving simultaneous equations. For example, when subtracting $-5x$ from $4x$, students need to remember to apply the rules for subtraction carefully. ($4x - (-5x) = 9x$)

Students use substitution when solving simultaneous equations that are not both linear. They should draw rough sketches of the graphs when possible so that they can check that their answers to the simultaneous equations make logical sense.

Students often solve for one variable and then forget to use substitution to find the value of the other variable.

When answering a question in an exercise or test or exam, remind students that they should reread the question to make sure that their answer is complete.

When solving an equation that contains algebraic fractions, students must remember to check whether their answers are valid. If the denominator is 0, the answer is undefined and must be rejected.

Preparation

Ensure students have graph paper, pencils and rulers for drawing straight lines.

Prepare a Cartesian plane showing the x- and y-axes and the origin.

Prepare solution charts of simultaneous linear and quadratic functions.

Prepare posters for the classroom that illustrate:

- two intersecting lines
- a line that intersects a parabola or a circle in two points
- a line that touches a parabola; the line is a tangent to the parabola and there will be only one solution with two equal roots
- a line that does not touch a parabola; this will illustrate no real solution to the simultaneous equations.

Introducing students to the topic

Revise basic methods to solve equations using mathematical operations (+, −, × and ÷). When there are two variables to solve, students need two equations.

Make sure students understand that *simultaneously* means *happen at the same time*. Encourage students to use the substitution method to solve all simultaneous equations. They need to write an equation with only one variable to solve. Students need to use substitution when the equations are not both linear.

Work through the examples with the class and then ask the students to work through Exercise 6.1.

Answers

Exercise 6.1 (page 78)

1. a) $x = 2$ and $y = 0$ b) $x = 3$ and $y = -2$
   c) $x = 8$ and $y = -10$ d) $x = 3$ and $y = 1$
   e) $x = 3$ and $y = 1$ f) $x = 3$ and $y = 7$
   g) $x = 1$ and $y = -2$ h) $x = 3$ and $y = -2$
   i) $x = 4$ and $y = -\frac{1}{2}$ j) $x = 3$ and $y = -2$

2. a) $\left(-\frac{6}{5}; \frac{13}{5}\right)$ b) $\left(\frac{20}{7}; \frac{9}{7}\right)$

Exercise 6.2 (page 80)

1. a) (2; 3) and (−2; −1) b) (0; −3) and $\left(\frac{3}{2}; 0\right)$
   c) (20; 8) and (12; 4) d) (1; 3) and (4; 0)
   e) (4; 6) and (−1; −4) f) $\left(0; \frac{15}{4}\right)$ and (−3; 0)
   g) (1; −5) and (5; −1) h) (0; −3) and (−3; 0)
   i) (−9; 144) and (8; −60) j) $\left(-3; \frac{1}{2}\right)$ and (4; 4)

2. a) (1; 4) and (−1; 6) b) (2; −3) and $\left(\frac{1}{2}; -\frac{9}{2}\right)$
   c) Use the quadratic formula: (−1.79; −1.79) and (2.79; 2.79)
   d) A(0; 13) and B(10.4; −7.8)
   e) A(−5; 0) and P(3; 4)

Exercise 6.3 (page 84)

1. He is 40 years old.
2. $b = 12$ m and $h = 7$ m
3. 6 and 22
4. 6 kg of apples and 4 kg of oranges
5. $x = 500$ km/h
6. Twelve ₦5 notes, eight ₦10 notes and four ₦50 notes
7. ₦40 000 at 7% and ₦60 000 at 5%
8. He sold 18 chickens and bought 30 chickens.

Assess your progress (page 85)

1. a) $x = 3$ and $y = -4$ b) $x = 5$ and $y = 11$
   c) $x = -2$ and $y = -4$ d) $x = 3$ and $y = -1$
   e) $x = \frac{1}{2}$ and $y = 1$ f) (4; 1) and (1; 4)
   g) (0; −6) and $\left(-\frac{7}{2}; -\frac{5}{2}\right)$ h) (4; −10) and (2; −6)

2. **a)** $\left(-\frac{8}{7}; -\frac{25}{7}\right)$ **b)** S(−2; −5) and R(5; 2)
c) B(2; 5) and A(−3; 0)

3. $\frac{x}{y} = -2$ or $\frac{x}{y} = -3$
$x = 16$ and $y = -8$ or $x = 12$ and $y = -4$

4. The graphs do not intersect.

5. $x = 2$ and $y = -6$

6. **a)** More than 200 cars **b)** 250 cars

7. ₦70 000 bonds and ₦80 000 stocks

8. **a)** 24 tiles
b) 240 tiles

9. Speed = 40 km/h

10. **a)** $x - y = \frac{15}{2} = 7.5$
$x + y = 10$
b) $x = 8.75$ and $y = 1.25$
c) $T = 6 \div 8.75 = \frac{24}{35}$ h = 41 min.

TOPIC 7 Surface area and volume of spheres

Introduction

In this topic, students will revise the surface area and volume of solids, revise compound shapes and frustums, calculate the total surface area of spheres and hemispheres and calculate the volume of spheres and hemispheres.

Revise three-dimensional (3-D) objects, total surface area (TSA), the units we use to measure area (such as square metres (m^2)), how to calculate the area of solids (such as cubes, cuboids, pyramids, cylinders, cones and prisms) and the terms faces, edges and vertices with the class. Also revise volume (the amount of space occupied by a solid) and the formulae we use to calculate the volume of solids (such as cubes, cuboids, pyramids, cylinders, cones, spheres and prisms). Also revise how to calculate the total surface area and volume of compound solids and frustums and the method we use to calculate the curved surface area and the volume of a truncated cone.

This year students will use formulae to calculate the surface area and volume of spheres and hemispheres. Students will also learn how to calculate the surface area and volume of a truncated pyramid.

Common difficulties

Students need to know the difference between faces, edges and vertices in objects. They need to be able to recognise and name the different objects they will study. Students need to learn all the formulae for surface area and volume and know which formula to use for each shape.

They need to read the questions carefully and decide whether they are being asked to calculate surface area or volume. Then students need to use the correct formulae for calculations.

It is important that students ensure that all measurements in a calculation are in the same unit. They also need to be able to convert units so that all measurements will be in the same unit.

Preparation

Prepare a poster with the different objects and a table with all the necessary formulae for surface area and volume.

Find objects with interesting shapes to illustrate how the work students do in this topic is useful in everyday life. Examples of such objects include cans, boxes and spheres of different sizes from a grocery store or a kitchen. Take a few solids apart to show students that the nets of objects are made up of two-dimensional shapes. For example, a cylinder is made up of two circles and a rectangle.

Introducing students to the topic

Revise surface area and volume with the students. Check that students know all the terms they will need when discussing area and volume and that they know how to use the different formulae.

Make sure that all students know the symbols and abbreviations on page 87 in the Student's Book.

Tell students that they will learn about spheres and hemispheres this year. When calculating area and volume that involve circles, students can use $\pi = \frac{22}{7}$ or $\pi = 3.14$ as a constant.

Work through the examples and ask students to complete Exercise 7.1.

Make sure the students understand and know the formulae on pages 89, 90, 92, 93, 95 and 97 in the Student's Book.

You can use the assessment exercise at the end of the topic for revision.

Answers

Exercise 7.1 (page 91)

1. Area = $2(3.2 \times 2.8) + 2(3.2 \times 9.5) + 2(2.8 \times 9.5)$
$= 131.92 \text{ cm}^2$

2. Height of triangle = $5\sqrt{3}$ (Pythagoras' theorem)
Area = $\frac{1}{2}(10 \times 5\sqrt{3}) \times 2 + 3(10 \times 15) = 536.6 \text{ cm}^2$

3. $85 = 25 + 4\left(\frac{1}{2} \times 5 \times s\right)$
$\therefore s = 6$ cm

4. Hypotenuse of triangle $= \sqrt{5^2 + 4.5^2}$ (Pythagoras' theorem)
$= 6.73$ cm
Area $= 2\left(\frac{1}{2} \times 4.5 \times 5\right) + (8 \times 4.5) + (8 \times 5) + (8 \times 6.73)$
$= 152.34 \text{ cm}^2$

5. $6s^2 = 15.36$
$\therefore s = 1.6$ cm

Exercise 7.2 (page 94)

1. Area $= 2 \times \frac{22}{7} \times (2.1)^2 + 2 \times \frac{22}{7} \times (2.1)(11) = 172.92 \text{ cm}^2$

2. **a)** Area $= \frac{22}{7} \times 8^2 + \frac{22}{7} \times 8 \times 17 = 628.57 \text{ cm}^2$
b) $s = 25$ cm (Pythagoras' theorem)
So, area $= \frac{22}{7} \times 7^2 + \frac{22}{7} \times 7 \times 25 = 704 \text{ cm}^2$

3. **a)** Area $= 4 \times \frac{22}{7} \times 8^2 = 804.57 \text{ cm}^2$
b) Area $= 4 \times \frac{22}{7} \times 4.6^2 = 266.01 \text{ cm}^2$
c) Area $= 4 \times \frac{22}{7} \times 6^2 = 452.57 \text{ cm}^2$
d) Area $= 4 \times \frac{22}{7} \times 7.6^2 = 726.13 \text{ cm}^2$

4. **a)** Area $= 3 \times \frac{22}{7} \times 9^2 = 763.71 \text{ cm}^2$
b) Area $= 3 \times \frac{22}{7} \times 8.2^2 = 633.98 \text{ cm}^2$
c) Area $= 3 \times \frac{22}{7} \times 8^2 = 603.43 \text{ cm}^2$
d) Area $= 3 \times \frac{22}{7} \times 9.4^2 = 833.11 \text{ cm}^2$

5. Area of cylinder A $= 2 \times \frac{22}{7} \times 5^2 + 2 \times \frac{22}{7} \times 5 \times 20$
$= 785.71 \text{ cm}^2$
Area of cylinder B $= 2 \times \frac{22}{7} \times 2.5^2 + 2 \times \frac{22}{7} \times 2.5 \times 40$
$= 667.86 \text{ cm}^2$
So, cylinder A has the larger surface area.

Exercise 7.3 (page 96)

1. Volume $= 5.1 \times 4.8 \times 12 = 293.76 \text{ cm}^3$
2. Height of triangle $= 5\sqrt{3}$
Volume $= \frac{1}{2} \times 10 \times 5\sqrt{3} \times 16 = 692.82 \text{ cm}^3$

3. $85 = \frac{1}{3}(5 \times 5) \times h$
$\therefore h = 10.2$ cm

$\therefore s = \sqrt{10.2^2 + 5^2}$ (Pythagoras' theorem)
$= 11.36$ cm

4. Volume $= \frac{1}{2}(30 \times 40) \times 60 = 36\ 000$ cm^3

5. $s^3 = 216$
$s = 6$ cm

6. Volume $= (20 \times 40 \times 15) - (16 \times 36 \times 13) = 4\ 512$ cm^3

Exercise 7.4 (page 98)

1. Volume $= \frac{22}{7} \times 1.6^2 \times 15 = 120.69$ cm^3

2. Volume $= \frac{1}{3} \times \frac{22}{7} \times 3^2 \times 8 = 75.43$ cm^3

3. Volume $= \frac{22}{7} \times 35^2 \times 75 = 288\ 750$ cm^3

4. Volume $= \frac{1}{3} \times \frac{22}{7} \times 2.5^2 \times 8 = 52.38$ cm^3

5 **a)** Volume $= \frac{4}{3} \times \frac{22}{7} \times 9^3 = 3\ 054.86$ cm^3
b) Volume $= \frac{4}{3} \times \frac{22}{7} \times 2.3^3 = 60$ cm^3
c) Volume $= \frac{4}{3} \times \frac{22}{7} \times 8^3 = 2\ 145.52$ cm^3
d) Volume $= \frac{4}{3} \times \frac{22}{7} \times 10.2^3 = 4\ 446.97$ cm^3

6. **a)** Volume $= \frac{2}{3} \times \frac{22}{7} \times 12^3 = 3\ 620.57$ cm^3
b) Volume $= \frac{2}{3} \times \frac{22}{7} \times 6.1^3 = 475.58$ cm^3
c) Volume $= \frac{2}{3} \times \frac{22}{7} \times 24^3 = 28\ 964.57$ cm^3
d) Volume $= \frac{2}{3} \times \frac{22}{7} \times 8.2^3 = 1\ 155.25$ cm^3

7. Volume cylinder A $= \frac{22}{7} \times 5^2 \times 20 = 1\ 571.43$ cm^3
Volume cylinder B $= \frac{22}{7} \times 2.5^2 \times 40 = 785.71$ cm^3
Cylinder A has the greater volume.

Exercise 7.5 (page 102)

1. Volume $= \frac{22}{7} \times 5^2 \times 90 + \frac{1}{3} \times \frac{22}{7} \times 5^2 \times 10 = 7\ 333.33$ mm^3
Slant height $= \sqrt{10^2 + 5^2}$ (Pythagoras' theorem)
$= 5\sqrt{5}$ mm
Total surface area
$= \left(\frac{22}{7} \times 5^2\right) + 2 \times \frac{22}{7} \times 5 \times 90) + \left(\frac{22}{7} \times 5 \times 5\sqrt{5}\right)$
$= 3\ 082.84$ mm^2

2. **a)** Volume = (2 × 1 × 1) + (5 × 1 × 1) = 7 m^3

Total surface area

= 3(2 × 1) + 2(1 × 1) + 3(5 × 1) + 2(1 × 1) + (3 × 1)

= 28 m^2

b) Perpendicular height = 12 cm (Pythagoras' theorem)

Volume = $\frac{1}{3} \times \frac{22}{7} \times 5^2 \times 12 + \frac{2}{3} \times \frac{22}{7} \times 5^3 = 576.19$ cm^3

Total surface area = $2 \times \frac{22}{7} \times 5^2 + \frac{22}{7} \times 5 \times 13$

= 361.43 cm^2

c) Volume = $\frac{22}{7} \times 10^2 \times 79 + \frac{2}{3} \times \frac{22}{7} \times 10^3$

= 26 923.81 cm^3

Total surface area

= $\left(\frac{22}{7} \times 10^2\right) + \left(2 \times \frac{22}{7} \times 10 \times 79\right) + \left(2 \times \frac{22}{7} \times 10^2\right)$

= 5 908.57 cm^2

d) Volume = volume of hemisphere + volume of cylinder + volume of cone = $\frac{2}{3}\pi r^3 + \pi r^2 h + \frac{1}{3}\pi r^2 h$

Height of the cone = 21 cm (Pythagoras' theorem)

So, height of cylinder = 80 − 20 − 21 = 39 cm

Volume = $\frac{2}{3}(\frac{22}{7})(20)^3 + \frac{22}{7}(20)^2(39) + \frac{1}{3}(\frac{22}{7})(20)^2(21)$

= 16 761.9 + 49 028.57 + 8 800

= 74 590.47 cm^3

Total surface area

= surface area hemisphere + surface area cylinder + surface area cone

= $2\pi r^2 + 2\pi rh + \pi rs$

= $2 \times \frac{22}{7} \times 20^2 + 2 \times \frac{22}{7} \times (20)(39) + \frac{22}{7}(20)(29)$

= 9 240 cm^2

3. Volume = volume hemisphere + volume cone

= $\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$

= $\frac{2}{3}(\frac{22}{7})(5)^3 + \frac{1}{3}(\frac{22}{7})(5)^2(12) = 576.19$ cm^3

Slant height = 13 cm (Pythagoras' theorem)

Total surface area

= surface area hemisphere + surface area cone

= $2\pi r^2 + \pi rs$

= $2 \times \frac{22}{7} \times 5^2 + \frac{22}{7}(5)(13)$

= 361.43 cm^2

4. **a)** Volume of cylinder = $\frac{22}{7} \times 8^2 \times 16 = 3\,218.29$ cm^3

b) Volume of ball = $\frac{4}{3}(\frac{22}{7})8^3 = 2\,145.52$ cm^3

c) Volume of space not taken up by the ball
$= 3\,218.29 - 2\,145.52 = 1\,072.77$ cm^3

5. **a)** Let the vertical height of the missing top of the cone be x.

Then, $\frac{x}{x + 12} - \frac{7.2}{12.6}$ (Similar triangles)

The ratio $\frac{x}{x + 12} = \frac{4}{7}$

$\therefore 7x = 4x + 48$

$x = 16$ cm

Slope height of small cone

$= \sqrt{16^2 + 7.2^2} = 17.54$ cm (Pythagoras' theorem)

Slope height of whole cone

$= \sqrt{28^2 + 12.6^2} = 30.7$ cm (Pythagoras' theorem)

Curved surface area of frustum

$= \frac{22}{7}(12.6)(30.7) - \frac{22}{7}(7.2)(17.54)$

$= 818.81$ cm^2

b) Volume of frustum

$= \frac{1}{3}(\frac{22}{7})(12.6)^2(28) - \frac{1}{3}(\frac{22}{7})(7.2)^2(16)$

$= 4\,656.96 - 868.94$

$= 3\,788.02$ cm^3

6. Let height of missing top pyramid be x.

$\therefore \frac{x + 10}{x} = \frac{6}{2} = 3$ (Proportion)

$\therefore 3x = x + 10$

$\therefore x = 5$ cm

Volume of frustum: $\frac{1}{3}(6 \times 4.5 \times 15) - \frac{1}{3}(2 \times 1.5 \times 5)$

$= 135 - 5$

$= 130$ cm^3

7. Slant height of whole cone

$= 75$ cm (Pythagoras' theorem)

a) Ratio of sides of similar triangles: 3 : 1

Slope height of small cone: 25 cm

Slope height of frustum: 50 cm

Curved surface area of frustum

$= \frac{22}{7}(21)(75) - \frac{22}{7}(7)(25)$

$= 4\,400$ cm^2

Total area of frustum $= 4\,400 + (\frac{22}{7})(21)^2 + (\frac{22}{7})(7)^2$

$= 5\,940$ cm^2

b) Volume of frustum $= \frac{1}{3}(\frac{22}{7})(21)^2(72) - \frac{1}{3}(\frac{22}{7})(7)^2(24)$
$= 33\ 264 - 1\ 232$
$= 32\ 032 \text{ cm}^3$

8. Height of top pyramid: 2 cm (Similarity)
Volume $= \frac{1}{3}(12^2 \times 8) - \frac{1}{3}(3^2 \times 2) = 378 \text{ cm}^3$

Assess your progress (page 104)

1. Total surface area $= 2(5 \times 12) + 2(12 \times 30) + 2(5 \times 30)$
$= 1\ 140 \text{ cm}^2$
Volume $= 5 \times 12 \times 30 = 1\ 800 \text{ cm}^3$

2. Total surface area
$= 2(\frac{1}{2} \times 30 \times 40) + (60 \times 40) + (30 \times 60) + (50 \times 60)$
$= 8\ 400 \text{ cm}^2$

3. Volume $= \frac{2}{3}(\frac{22}{7})(3)^3 + (\frac{22}{7} \times 3^2 \times 10) = 339.43 \text{ m}^3$

4. **a)** Volume
= volume of hemisphere + volume of cylinder
+ volume of cone
$= \frac{2}{3}\pi r^3 + \pi r^2 h + \frac{1}{3}\pi r^2 h$
Radius = 6 cm (Pythagoras' theorem)
So, height of cylinder = 18 − 6 − 8 = 4 cm
Volume $= \frac{2}{3}(\frac{22}{7})(6)^3 + \frac{22}{7}(6)^2(12) + \frac{1}{3}(\frac{22}{7})(6)^2(8)$
$= 452.57 + 1\ 357.71 + 301.71$
$= 2\ 112 \text{ cm}^3$
Total surface area
= surface area hemisphere + surface area cylinder
+ surface area cone
$= 2\pi r^2 + 2\pi rh + \pi rs$
$= 2 \times \frac{22}{7} \times 6^2 + 2 \times \frac{22}{7} \times 6(4) + \frac{22}{7}(6)(10)$
$= 565.72 \text{ cm}^2$

b) Volume $= \frac{2}{3}(\frac{22}{7})(5)^3 + \frac{22}{7}(5)^2(6) = 733.3 \text{ m}^3$
Total surface area
= surface area hemisphere + surface area cylinder
$= 2\pi r^2 + (2\pi rh + \pi r^2)$
$= 2 \times \frac{22}{7} \times 5^2 + [2 \times \frac{22}{7} \times (5)(6) + \frac{22}{7} \times 5^2]$
$= 424.28 \text{ m}^2$

5. Volume $= \frac{22}{7}(9)^2(45) - \frac{22}{7}(6)^2(45) = 6\ 364.3 \text{ cm}^3$

6. Slant height = 10.77 (Pythagoras' theorem)
Total surface area
= surface area cylinder + surface area cone
$= 2\pi rh + \pi r^2 + \pi rs$
$= \left[2 \times \frac{22}{7} \times (4)(12)\right] + \left[\frac{22}{7} \times 4^2\right] + \left[\frac{22}{7}(4)(10.77)\right]$
$= 487.39\ \text{m}^2$

7. Volume $= \frac{22}{7}(30)^2(40) + \frac{1}{3}(\frac{22}{7})(30)^2(15) = 127\ 285.71\ \text{cm}^3$

8. **a)** Total surface area
$= (5 \times 10) + 2(1 \times 10) + 2(5 \times 1) + 2(2 \times 10)$
$+ 2(3 \times 10) + (1 \times 10) + 2(1 \times 3)$
$= 196\ \text{cm}^2$
Volume $= (5 \times 10 \times 1) + (3 \times 10 \times 1) = 80\ \text{cm}^3$
b) Slant height $= \sqrt{10} = 3.16$ cm (Pythagoras' theorem)
Total surface area $= (2 \times 2) + 4(\frac{1}{2} \times 2 \times 3.16)$
$= 16.64\ \text{cm}^2$
Volume $= \frac{1}{3}(2 \times 2) \times 3 = 4\ \text{cm}^3$
c) Hypotenuse = 2.6 m
Total surface area
$= 5(1.5)^2 + 2(1.5 \times 2.6) + 2(\frac{1}{2} \times 1.5 \times 2.5) = 22.8\ \text{m}^2$
Volume $= 1.5^3 + \frac{1}{2} \times 1.5 \times 2.5 \times 1.5 = 6.19\ \text{m}^3$

9. **a)** Height of missing cone: 4 cm (Similarity ratio: 3 : 1)
So, volume $= \frac{1}{3}(\frac{22}{7})(7.5)^2(12) - \frac{1}{3}(\frac{22}{7})(2.5)^2(4)$
$= 680.95\ \text{cm}^3$
b) Height of missing pyramid
= 18 cm (Similarity ratio: 2 : 1)
So, volume $= \frac{1}{3}(30 \times 12 \times 36) - \frac{1}{3}(15 \times 6 \times 18)$
$= 3\ 780\ \text{cm}^3$

10. **a)** Volume $= \frac{4}{3}(\frac{22}{7})(21.5)^3 = 41\ 646.5\ \text{mm}^3$
b) Volume $= \frac{4}{3}(\frac{22}{7})(3.3)^3 = 150.6\ \text{cm}^3$
c) $2 \times \frac{22}{7} \times r = 225$ mm
So, $r = 35.8$ mm
So, volume $= \frac{4}{3}(\frac{22}{7})(35.8)^3 = 192\ 270.4\ \text{mm}^3$
d) Volume $= \frac{4}{3}(\frac{22}{7})(11)^3 = 5\ 577.5\ \text{cm}^3$
e) $2 \times \frac{22}{7} \times r = 75$ cm
So, $r = 11.9$ cm
So, volume $= \frac{4}{3}(\frac{22}{7})(11.9)^3 = 7\ 061.6\ \text{cm}^3$

TOPIC 8 Longitude and latitude

Introduction

In this topic, you will introduce your students to lines of longitude and latitude for the first time. Some of your students may be familiar with these concepts, while others will learn about them this year for the first time.

In the first part of this topic, your students will realise that the earth is almost a perfect sphere. They will learn about a few of the most important features of the earth's geography. They will identify and locate these features on a skeletal globe and on a real globe. Students will learn about lines of longitude and latitude and identify and interpret these lines on a map. They will also interpret the coordinates of longitude and latitude of a few of the world's capital cities.

In the second section in this topic, your students will revise the geometry of circles and spheres, before they tackle a variety of calculations. They will calculate the length of a parallel of latitude and its radius, the distance between two points on the earth's surface and the difference in the times of two places.

Common difficulties

Students often confuse lines of longitude and lines of latitude and also the coordinates of latitude and longitude. Refer students who struggle to the posters that you have prepared.

Students need to become comfortable with complex multi-step calculations on their calculators. When using a calculator, your students need to know how to recognise the degree mode on the model of the calculator they use. If their calculators are not in degree mode, all their trigonometric calculations will be incorrect. So, they need to be able to set their calculators to this mode. You will need to remediate this for every different calculator model.

If students do not have a calculator, they need to practise their skills at using the log tables on pages 274 to 277 of the Student's Book.

Preparation

Prepare the following visual aids to use in the classroom:

- Examples of circles and spheres
- Real globes (you will need at least one in the classroom; it would be useful to have more than one globe for students to use when they work in groups in Exercise 8.1)
- Skeletal globes (this is an optional resource, as we provide a diagram of a skeletal globe in Exercise 8.1.)
- Maps of Nigeria, Africa and world maps will help to enhance your students' understanding of, and participation in, this topic.

Prepare the following posters to display in the classroom:

- a poster that shows the four main cardinal points of the compass (north, south, east and west)
- a poster that shows a circle with the main parts of a circle labelled; see the example on page 114 in the Student's Book
- a poster that shows examples of coordinate pairs such as (50° N, 35° E), (10° S, 35° E) and (67° S, 132° W); label each example clearly, so that your students can refer to this poster when they are not sure which coordinate describes latitude and which one describes longitude
- a poster that shows the different formulae that are used in this topic.

Introducing students to the topic

Write the following terms on the board without any explanation:

- North Pole
- South Pole
- equator
- northern hemisphere
- southern hemisphere
- Greenwich meridian
- International Date Line.

Explain to your class that these terms all relate to the geography of the earth. Point to each one in turn and ask your class to put up their hands if they have heard the term before. Each time, ask a volunteer to explain the term to the class.

This interaction will allow you to discover how familiar your students already are with a few of the main features of the earth's geography.

Answers

Exercise 8.1 (page 112)

1. **a)** A **b)** E **c)** C
   d) B, C and D **e)** F, G and H **f)** C, F, G and H
   g) B and D

2. Students work in small groups and identify the following on a globe.
   a) the North Pole **b)** the South Pole
   c) the equator **d)** lines of latitude
   e) lines of longitude **f)** great circles
   g) small circles

3. **a)** Nigeria lies in the Northern hemisphere. On the map, the lines of latitude are all marked N. This shows us that Nigeria lies in the northern hemisphere.
   b) Kaduna **c)** Hadejia
   d) Port Harcourt **e)** Sokoto
   f) Ikeja and Abeokuta **g)** Maiduguri
   h) Uyo and Calabar **i)** Lagos and Maiduguri

4. **a)** Abuja, Beijing, Cairo, London, Mexico City, New Delhi, Paris, Porto Novo, Reykjavik, Rome, Washington DC
   b) Lima, Luanda, Nairobi, Pretoria, Wellington
   c) Abuja, Beijing, Cairo, Luanda, Nairobi, New Delhi, Paris, Porto Novo, Pretoria, Rome, Wellington
   d) Lima, London, Mexico City, Reykjavik, Washington DC

Exercise 8.2 (page 115)

1. **a)** $C = 2\pi r = 2 \times 3.142 \times 25 = 157.1$ cm
   b) $C = 2\pi r = 2 \times 3.142 \times 3 = 18.9$ km

2. **a)** $C = 2\pi r$
   $\therefore r = \frac{C}{2\pi}$
   $= \frac{168}{(2 \times 3.142)}$
   $= 26.7$ m
   b) $r = \frac{C}{2\pi} = \frac{43}{(2 \times 3.142)} = 6.8$ cm

3. Arc length $= 2\pi r \times \frac{\theta}{360°} = 2 \times 3.142 \times 12 \times \frac{55°}{360°} = 11.5$ cm

4. Arc length $= 2\pi r \times \frac{\theta}{360°}$

$$\therefore r = \frac{\text{arc length}}{2\pi} \times \frac{360°}{\theta}$$
$$= \frac{85}{(2 \times 3.142)} \times \frac{360°}{120°}$$
$$= 40.6 \text{ mm}$$

5. a) $\frac{98}{2} = 49$ mm

   Arc EF is shorter than half of the circumference of the circle, so arc EF is a minor arc.

   b) Arc length $= 2\pi r \times \frac{\theta}{360°}$

   $2\pi r$ = the circumference of the circle = 98 mm

   $98 \times \frac{\theta}{360°} = 31$

   $\therefore \theta = \frac{31}{98} \times 360°$

   $= 113.9°$

   c) $360° - 113.9° = 246.1°$

   So, the major arc EF subtends an angle of 246.1° at the centre of the circle.

6. $C = 2\pi r$

   $\therefore r = \frac{C}{2\pi}$

   $= \frac{106.84}{(2 \times 3.142)}$

   $= 17$ cm

   The radius of the circle is smaller than the radius of the sphere, so the circle is a small circle of the sphere and not a great circle.

Exercise 8.3 (page 117)

1. a) $r = 6\,400 \times \cos\theta = 6\,400 \times \cos 15° = 6\,182$ km
   b) $r = 6\,400 \times \cos\theta = 6\,400 \times \cos 60° = 3\,200$ km
   c) $r = 6\,400 \times \cos\theta = 6\,400 \times \cos 85° = 558$ km
   d) $r = 6\,400 \times \cos\theta = 6\,400 \times \cos 43° = 4\,681$ km
   e) $r = 6\,400 \times \cos\theta = 6\,400 \times \cos 28° = 5\,651$ km
   f) $r = 6\,400 \times \cos\theta = 6\,400 \times \cos 75° = 1\,656$ km

2. a) Length of the parallel of latitude (15° N)

   $= 2 \times \pi \times r$

   $= 2 \times 3.142 \times 6\,182$

   $= 38\,848$ km

b) Length of the parallel of latitude (60° S)
$= 2 \times \pi \times r$
$= 2 \times 3.142 \times 3\,200$
$= 20\,109$ km

c) Length of the parallel of latitude (85° N)
$= 2 \times \pi \times r$
$= 2 \times 3.142 \times 558$
$= 3\,506$ km

d) Length of the parallel of latitude (43° S)
$= 2 \times \pi \times r$
$= 2 \times 3.142 \times 4\,681$
$= 29\,415$ km

e) Length of the parallel of latitude (28° N)
$= 2 \times \pi \times r$
$= 2 \times 3.142 \times 5\,651$
$= 35\,511$ km

f) Length of the parallel of latitude (75° S)
$= 2 \times \pi \times r$
$= 2 \times 3.142 \times 1\,656$
$= 10\,406$ km

3. 60° N and 60° S; students may have noticed this from the answer to question 1(c).
Or, they could set up an equation and solve for θ:
$6\,400 \times \cos\theta = \frac{1}{2} \times 6\,400$
$\therefore \cos\theta = \frac{1}{2}$
$\therefore \theta = 60°$
There will be two solutions − one in the northern hemisphere and one in the southern hemisphere.

Exercise 8.4 (page 119)

1. All answers are rounded off to the nearest kilometre.

a) Angle subtended by arc AB at the centre of the earth
$= 82° - 35° = 47°$
$\therefore$ Arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{47°}{360°}$
$= 5\,251$ km

b) Angle subtended by arc AB at the centre of the earth
$= 62° - 6° = 56°$
$\therefore$ Arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{56°}{360°}$
$= 6\,256$ km

c) Angle subtended by arc AB at the centre of the earth
$= 73° + 47° = 120°$
$\therefore$ Arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{120°}{360°}$
$= 13\,406$ km

d) Angle subtended by arc AB at the centre of the earth
$= 22° + 64° = 86°$
$\therefore$ Arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{86°}{360°}$
$= 9\,608$ km

e) Angle subtended by arc AB at the centre of the earth
$= 49° - 39° = 10°$
$\therefore$ Arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{10°}{360°}$
$= 1\,117$ km

f) Angle subtended by arc AB at the centre of the earth
$= 14° + 55° = 69°$
$\therefore$ Arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{69°}{360°}$
$= 7\,708$ km

2. All answers are rounded off to the nearest kilometre, where applicable.

a) Angle subtended by the arc between the two cities at the centre of the earth $= 22.5° + 48° = 70.5°$
$\therefore$ Distance $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{70.5°}{360°}$
$= 7\,876$ km

b) Angle subtended by the arc between the two cities at the centre of the earth $= 44° + 34° = 78°$
$\therefore$ Distance $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{78°}{360°}$
$= 8\,714$ km

c) Angle subtended by the arc between the two cities at the centre of the earth $= 44.5° + 29° = 73.5°$
$\therefore$ Distance $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{73.5°}{360°}$
$= 8\,211$ km

d) Angle subtended by the arc between the two cities at the centre of the earth $= 51° - 6.5° = 44.5°$
$\therefore$ Distance $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{44.5°}{360°}$
$= 4\,971$ km

e) Angle subtended by the arc between the two cities at the centre of the earth $= 43.5° - 6.5° = 37°$
$\therefore$ Distance $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{37°}{360°}$
$= 4\,133$ km

f) Angle subtended by the arc between the two cities at the centre of the earth = 43.5° − 5° = 38.5°

$\therefore$ Distance = $2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{38.5°}{360°}$

$= 4\,301$ km

3. The point furthest from point P is the point on the opposite side of the great circle through P. So, the coordinates of that point:

(10° N, 180° − 115° W) = (10° N, 65° W)

Exercise 8.5 (page 123)

Distances given to nearest kilometre, where applicable.

1. a) Let S represent the South Pole.

Then, $A\hat{S}B = 113° - 76° = 37°$

To calculate the radius of the small circle of the line of latitude (78° S):

$r = 6\,400 \times \cos 78° = 1\,331$ km

Arc AB = $2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 1\,331 \times \frac{37°}{360°}$

$= 860$ km

b) Let N represent the North Pole.

Then, $A\hat{N}B = 103° - 57° = 46°$

To calculate the radius of the small circle of the line of latitude (54° N):

$r = 6\,400 \times \cos 54° = 3\,762$ km

Arc AB = $2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 3\,762 \times \frac{46°}{360°}$

$= 3\,021$ km

c) Let N represent the North Pole.

Then, $A\hat{N}B = 73° + 42° = 115°$

To calculate the radius of the small circle of the line of latitude (37° N):

$r = 6\,400 \times \cos 37° = 5\,111$ km

Arc AB = $2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 5\,111 \times \frac{115°}{360°}$

$= 10\,260$ km

d) Let S represent the South Pole.

Then, $A\hat{S}B = 149° + 36° = 185°$

This angle is greater than 180°, so it is a major arc. Minor arc AB will give the shortest distance between A and B along the surface of the earth and it will make an angle of (360° − 185°) = 175° with the South Pole.

To calculate the radius of the small circle of the line of latitude (80° S):

$r = 6\,400 \times \cos 80° = 1\,111$ km

Arc AB $= 2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 1\,111 \times \frac{175°}{360°}$

$= 3\,394$ km

e) Let N represent the North Pole.

Then, $A\hat{N}B = 47° + 129° = 176°$

To calculate the radius of the small circle of the line of latitude (29° N):

$r = 6\,400 \times \cos 29° = 5\,598$ km

Arc AB $= 2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 5\,598 \times \frac{176°}{360°}$

$= 17\,198$ km

f) Let S represent the South Pole.

Then, $A\hat{S}B = 168° + 168° = 336°$

This angle is greater than 180°, so it is a major arc. Minor arc AB will give the shortest distance between A and B along the surface of the earth and it will make an angle of $(360° - 336°) = 24°$ with the South Pole.

To calculate the radius of the small circle of the line of latitude (15° S):

$r = 6\,400 \times \cos 15° = 6\,182$ km

Arc AB $= 2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 6\,182 \times \frac{24°}{360°}$

$= 2\,590$ km

2. **a)** Angle subtended by the arc between the two cities at the North Pole = 4.5° + 37.5° = 42°

To calculate the radius of the small circle of the line of latitude (56° N):

$r = 6\,400 \times \cos 56° = 3\,579$ km

Distance $= 2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 3\,579 \times \frac{42°}{360°}$

$= 2\,624$ km

b) Angle subtended by the arc between the two cities at the North Pole = 13.5° + 2° = 15.5°

To calculate the radius of the small circle of the line of latitude (52.5° N):

$r = 6\,400 \times \cos 52.5° = 3\,896$ km

Distance $= 2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 3\,896 \times \frac{15.5°}{360°}$

$= 1\,054$ km

c) Angle subtended by the arc between the two cities at the North Pole = 5.5° + 11° = 16.5°
To calculate the radius of the small circle of the line of latitude (6.5° N):
$r = 6\,400 \times \cos 6.5° = 6\,359$ km
Distance $= 2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 6\,359 \times \frac{16.5°}{360°}$
$= 1\,831$ km

d) Angle subtended by the arc between the two cities at the South Pole = 57.5° − 28.5° = 29°
To calculate the radius of the small circle of the line of latitude (20° S):
$r = 6\,400 \times \cos 20° = 6\,014$ km
Distance $= 2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 6\,014 \times \frac{29°}{360°}$
$= 3\,044$ km

e) Angle subtended by the arc between the two cities at the South Pole = 151° − 18.5° = 132.5°
To calculate the radius of the small circle of the line of latitude (34° S):
$r = 6\,400 \times \cos 34° = 5\,306$ km
Distance $= 2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 5\,306 \times \frac{132.5°}{360°}$
$= 12\,272$ km

3. Angle subtended by the arc between Abuja and Addis Ababa at the North Pole = 38.5° − 7.5° = 31°.
Angle subtended by the arc between Abuja and Panama City at the North Pole = 7.5° + 79.5° = 87°.
Angle subtended by the arc between Addis Ababa and Panama City at the North Pole = 38.5° + 79.5° = 118°.
Abuja and Addis Ababa are closest together and Addis Ababa and Panama City are furthest apart.
a) To calculate the radius of the small circle of the line of latitude (9° N):
$r = 6\,400 \times \cos 9° = 6\,321$ km
Abuja to Addis Ababa $= 2\pi r \times \frac{31°}{360°}$
$= 2 \times 3.142 \times 6\,321 \times \frac{31°}{360°}$
$= 3\,420$ km
b) Addis Ababa to Panama City
$= 2\pi r \times \frac{118°}{360°}$
$= 2 \times 3.142 \times 6\,321 \times \frac{118°}{360°}$
$= 13\,020$ km

Exercise 8.6 (page 123)

Lengths of time are given to the nearest 30 minutes.

1. a) Difference between the coordinates of longitude
= 77.0° + 7.4° = 84.4°
Time difference = $\frac{84.4°}{15°}$ hours = 5.5 hours

b) Difference between the coordinates of longitude
= 99.1° + 2.4° = 101.5°
Time difference = $\frac{101.5°}{15°}$ hours = 7 hours

c) Difference between the coordinates of longitude
= 21.8° + 31.2° = 53°
Time difference = $\frac{53°}{15°}$ hours = 3.5 hours

d) Difference between the coordinates of longitude
= 77.2° − 36.8° = 40.4°
Time difference = $\frac{40.4°}{15°}$ hours = 2.5 hours

e) Difference between the coordinates of longitude
= 116.4° − 2.6° = 113.8°
Time difference = $\frac{113.8°}{15°}$ hours = 7.5 hours

f) Difference between the coordinates of longitude
= 174.8° − 28.2° = 146.6°
Time difference = $\frac{146.6°}{15°}$ hours = 10 hours

g) Difference between the coordinates of longitude
= 77.0° + 13.3° = 90.3°
Time difference = $\frac{90.3°}{15°}$ hours = 6 hours

h) Difference between the coordinates of longitude
= 12.5° + 0.1° = 12.6°
Time difference = $\frac{12.6°}{15°}$ hours = 1 hour

2. a) Difference between the coordinates of longitude
= 7.4° − 0.0° = 7.4°
Time difference = $\frac{7.4°}{15°}$ hours = 0.5 hours
Abuja is further east than Greenwich, so it is 0.5 hours ahead of Greenwich. So, the time in Abuja is 12:30.

b) Difference between the coordinates of longitude
= 174.8° − 0.0° = 174.8°
Time difference = $\frac{174.8°}{15°}$ hours = 11.5 hours
Wellington is further east than Greenwich, so it is 11.5 hours ahead of Greenwich. So, the time in Wellington is 23:30.

c) Difference between the coordinates of longitude
= 36.8° − 0.0° = 38.8°
Time difference = $\frac{36.8°}{15°}$ hours = 2.5 hours
Nairobi is further east than Greenwich, so it is 2.5 hours ahead of Greenwich. So, the time in Nairobi is 14:30.

d) Difference between the coordinates of longitude
= 99.1° − 0.0° = 99.1°
Time difference = $\frac{99.1°}{15°}$ hours = 6.5 hours
Mexico City is further west than Greenwich, so it is 6.5 hours behind Greenwich. So, the time in Mexico City is 05:30.

e) Difference between the coordinates of longitude
= 116.4° − 0.0° = 116.4°
Time difference = $\frac{116.4°}{15°}$ hours = 8 hours
Beijing is further east than Greenwich, so it is 8 hours ahead of Greenwich. So, the time in Beijing is 20:00.

f) Difference between the coordinates of longitude
= 77° − 0.0° = 77°
Time difference = $\frac{77°}{15°}$ hours = 5 hours
Washington DC is further west than Greenwich, so it is 5 hours behind Greenwich. So, the time in Washington DC is 07:00.

Assess your progress (page 124)

1. The first coordinate (9.1° N) is the coordinate of latitude. This means that Ajuba lies 9.1° north of the equator. It also means that Ajuba lies in the northern hemisphere. The second coordinate (7.4° E) means that Ajuba lies 7.4° east of the Greenwich meridian.

2. Arc length = $2\pi r \times \frac{\theta}{360°} = 2 \times 3.142 \times 7.3 \times \frac{110°}{360°}$
$= 14.0$ cm

3. $C = 2\pi r = 2 \times 3.142 \times 360 = 2\ 262.2$ m

4. **a)** Angle subtended by the north–south line at the centre of the earth = 180°
So, distance = $2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\ 400 \times \frac{180°}{360°}$
$= 20\ 108.8$ km (to one decimal place)
Distance between the two poles: 20 109 km (to the nearest kilometre).

b) This distance is half the circumference of the earth.

5. **a)** Angle subtended by arc AB at the centre of the earth
$= 81° - 36° = 45°$
So, arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{45°}{360°}$
$= 5\,027$ km (to the nearest kilometre)

b) Angle subtended by arc AB at the centre of the earth
$= 85° + 50° = 135°$
So, arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{135°}{360°}$
$= 15\,082$ km (to the nearest kilometre)

c) The angle subtended by arc AB at the centre of the earth $= 54° + 75° = 129°$
So, arc AB $= 2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{129°}{360°}$
$= 14\,411$ km (to the nearest kilometre)

6. **a)** Let N represent the North Pole.
Then, $A\hat{N}B = 166° - 46° = 20°$
To calculate the radius of the small circle of the line of latitude (64° N):
$r = 6\,400 \times \cos 64°$
$= 2\,806$ km (to the nearest kilometre)
arc AB $= 2\pi r \times \frac{A\hat{N}B}{360°} = 2 \times 3.142 \times 2\,806 \times \frac{20°}{360°}$
$= 980$ km (to the nearest kilometre)

b) Let S represent the South Pole.
Then, $A\hat{S}B = 54° - 35° = 19°$
To calculate the radius of the small circle of the line of latitude (70° S):
$r = 6\,400 \times \cos 70°$
$= 2\,189$ km (to the nearest kilometre)
Arc AB $= 2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 2\,189 \times \frac{19°}{360°}$
$= 726$ km (to the nearest kilometre)

c) Let S represent the South Pole.
Then, $A\hat{S}B = 56° + 142° = 198°$
This angle is greater than 180°, so it is a major arc. The minor arc AB will give the shortest distance between A and B along the surface of the earth and it will make an angle of $(360° - 198°) = 162°$ with the South Pole.
To calculate the radius of the small circle of the line of latitude (89° S):

$r = 6\,400 \times \cos 89°$
$= 112$ km (to the nearest kilometre)
Arc AB $= 2\pi r \times \frac{A\hat{S}B}{360°} = 2 \times 3.142 \times 117 \times \frac{162°}{360°}$
$= 317$ km (to the nearest kilometre)

7. **a)** Difference between the coordinates of longitude
$= 151° - 5.5° = 145.5°$
Time difference $= \frac{145.5°}{15°}$ hours
$= 9.5$ hours (to the nearest 30 minutes)

b) Difference between the coordinates of longitude
$= 4.5° + 32.5° = 37°$
Time difference $= \frac{37°}{15°}$ hours
$= 2.5$ hours (to the nearest 30 minutes)

c) Difference between the coordinates of longitude
$= 122° - 11° = 111°$
Time difference $= \frac{111°}{15°}$ hours
$= 7.5$ hours (to the nearest 30 minutes)

8. **a)** Difference between the coordinates of longitude
$= 7° - 0.0° = 7°$
Time difference $= \frac{7°}{15°}$ hours
$= 0.5$ hours (to the nearest 30 minutes)
Port Harcourt is further east than Greenwich, so it is 0.5 hours ahead of Greenwich. So, the time in Port Harcourt is 00:30.

b) Difference between the coordinates of longitude
$= 114° - 0.0° = 114°$
Time difference $= \frac{114°}{15°}$ hours
$= 7.5$ hours (to the nearest 30 minutes)
Hong Kong is further east than Greenwich, so it is 7.5 hours ahead of Greenwich. So, the time in Hong Kong is 07:30.

c) Difference between the coordinates of longitude
$= 74° - 0.0° = 74°$
Time difference $= \frac{74°}{15°}$ hours
$= 5$ hours (to the nearest 30 minutes)
New York is further west than Greenwich, so it is 5 hours behind Greenwich. So, the time in New York is 19:00.

TOPIC 9

Finance

Introduction

In this topic, students will calculate simple interest, calculate compound interest, determine the depreciation value of an item, compute annuities, compute amortisation, calculate compound interest using logarithm tables, calculate interest on bonds and debentures and solve problems involving rates, taxes and VAT.

Preparation

Prepare charts to display in the classroom:

- formulae for calculating simple and compound increase and decrease
- formulae for computing annuities and amortisation with examples
- income tax rates and VAT
- logarithm changes of base law with examples
- sections from logarithm and antilogarithm tables.

Common difficulties

Students often use the incorrect formulae for financial calculations. When calculating simple interest, if it is only the interest that is required, use the formula, $SI = \frac{Prt}{100}$ and when the final amount needs to be calculated, use the formula, $A = P(1 \pm i \times n)$

It is important to understand the difference between simple interest and compound interest.

Students need to decide whether an amount increases (+) or decreases (−).

When one payment is made, they use the formulae: $A = P(1 \pm i \times n)$ for simple interest and the formula $A = P(1 \pm i)^n$ for compound interest.

When doing a financial calculation, it is good to write down all the known values to help you identify the values that you need to calculate. Students often confuse P (the principal or initial value) and A (the final amount).

When regular payments are made, use the annuity formulae:

- to calculate the future value of an annuity: $F_v = \frac{x[(1 + i)^n - 1}{i}$
- to calculate instalments for the bond or loan repayments: $P_v = \frac{x[1 - (1 + i)^{-n}]}{i}$.

Students need to understand the difference between the different types of tax.

Remind students that we do not include the unit (₦) in a calculation in order to simplify the calculation. We include the unit in the final answer.

Introducing students to the topic

Discuss the difference between simple and compound increase and decrease. Use simple examples to show how to find different variables in the formulae: $SI = \frac{Prt}{100}$, $A = P(1 \pm i \times n)$ and $A = P(1 \pm i)^n$. Make sure students know what all the letters in each formulae represent.

Explain how hire purchase works when purchasing expensive household items and give students an example to illustrate how expensive this form of purchasing goods is. Work through the examples in the Student's Book and ask the students to answer the questions in the exercises.

Work through the worked examples to remind students how to use the change of base rule to write logarithms with base 10. They can then use the logarithm tables for the necessary calculations.

Explain how to calculate regular payments when working with annuities and loans including bonds and debentures. Show students how to use the formulae to calculate the value of an annuity ($F_v = \frac{x[(1 + i)^n - 1}{i}$) and the present value of an annuity (or loan) ($P_v = \frac{x[1 - (1 + i)^{-n}]}{i}$).

The amortisation of a loan means working out the balance owing at any stage of the loan.

Explain the role of the different taxes that are levied by the government in order to pay for services such as healthcare and education.

Answers

Exercise 9.1 (page 129)

1. **a)** ₦36 000 **b)** ₦30 720 **c)** ₦22 050 **d)** ₦102 300

2. **a)** ₦100 300 **b)** ₦979 200 **c)** ₦115 200 **d)** ₦770 000

3. **a)** 6 years **b)** 3 years **c)** 7 years **d)** 5 years

4. **a)** 6 years **b)** 5 years **c)** 8 years **d)** 7.5 years

5. **a)** ₦150 000 **b)** ₦58 000 **c)** ₦250 000 **d)** ₦820 000

6. **a)** 9% **b)** 12% **c)** 8.5% **d)** 8%

7. Deposit = ₦8 500
85 000 − 8 500 = 76 500
$76\ 500(1 + 0.18 \times 2) = 104\ 040$
Monthly instalments: ₦4 335

8. ₦180 000

9. 15%

10. Interest rate: 12.5%

Exercise 9.2 (page 133)

1. **a)** ₦118 810 **b)** ₦112 568 **c)** ₦315 619 **d)** ₦590 070

2. **a)** ₦34 411 **b)** ₦68 106 **c)** ₦25 683 **d)** ₦586 660

3. **a)** 7 years **b)** 5 years **c)** 4 years **d)** 6 years

4. **a)** ₦14 500 **b)** ₦92 001.15
c) ₦260 067 **d)** ₦633 340

5. 4 years

6. Option A: ₦2 760 000; option B: ₦2 805 621
Option B: (11% compound interest for 6 years) is the better option.

7. ₦4 189 500

8. 6 years

9. **a)** 14% **b)** \$96

10. ₦3 540 039

Exercise 9.3 (page 137)

1. **a)** ₦38 203.20 **b)** ₦369 595.86
c) ₦827 566.80 **d)** ₦1 491 303.25

2. **a)** ₦242 718.45 **b)** ₦500 000
c) ₦523 505 **d)** ₦650 000

3. ₦200 000

4. **a)** ₦1 008 545 **b)** ₦6 101 425
c) ₦5 092 880 **d)** ₦850 982

5. **a)** ₦66 548
b) F_v of payments made for 5 years: 77 123
F_v of loan in 5 years: 119 922
So, the outstanding balance is ₦42 799.

6. **a)** ₦3 000 982.50
b) ₦2 320 000
c) ₦58 000 008 + 3 000 982.50 = ₦61 000 990.76

7. **a)** Annual repayment of the loan = ₦1 287 786.81
b) Balance outstanding on a loan
= money owing − money paid
Future value of the loan after two years:
$F_v = 2\ 800\ 000(1 + 0.18)^2 =$ ₦3 898 720
Amounts paid plus two years' interest accumulated on payment:
$F_v = \frac{1\ 287\ 786.81[(1 + 0.18)^2 - 1]}{0.18} = 2\ 807\ 375.25$
Balance of loan outstanding
= 3 898 720 − 2 807 375.25 = ₦1 091 344.75

8. 25 000 000 × 0.13 = 3 250 000
In three years, he earned 3 250 000 × 3 = ₦9 750 000

Exercise 9.4 (page 140)

1. **a)** ₦6 650
   b) 21 000 + 19 800 = ₦40 800
   c) 21 000 + 33 000 + 75 000 + 95 000 + 180 600
   = ₦404 600
   d) 21 000 + 33 000 + 75 000 + 95 000 + 176 000 + 72 000
   = ₦472 000

2. **a)** ₦1 160 000 **b)** ₦350 000
   c) ₦215 000 **d)** ₦271 000

3. **a)** ₦600 **b)** ₦2 250 **c)** ₦450 **d)** ₦3 428

4. **a)** ₦2 700 000 **b)** ₦2 950 000
   c) ₦50 550 **d)** ₦78 800

5. ₦9 095

6. **a)** ₦2 128 000
   b) Previous income tax
   = 21 000 + 33 000 + 75 000 + 95 000 + 63 000
   = 287 000
   Tax payable on increased income
   = 21 000 + 33 000 + 75 000 + 95 000 + 110 880
   = 334 880
   He pays ₦47 880 more income tax.
   Net salary increase = 228 000 − 47 880 = ₦180 120

7. **a)** $480\ 000(1.05)^5 = 612\ 615.15$
   b) $F_v = \frac{x[(1 + 0.07)^5 - 1]}{0.07}$
   $612\ 615.15 = \frac{x[(1 + 0.07)^5 - 1]}{0.07}$
   So, x = ₦106 528

Assess your progress (page 141)

1. **a)** ₦79 552 **b)** 6 years
   c) ₦890 006 **d)** ₦140 953

2. **a)** ₦1 457 500 **b)** ₦1 627 431

3. **a)** ₦44 250
   b) 250 750(1 + 0.22 × 2) = 361 080 ÷ 24 = ₦15 045

c) 361 080 + 44 250 = ₦405 330
A customer will pay ₦110 330 more.

4. **a)** $100\ 000(1 + 0.12)^5$ = ₦176 234
So, interest = ₦76 234

b) $F_v = \frac{20\ 000[1.12^5 - 1]}{0.12}$ = ₦127 057

5. $3\ 790\ 000(1 - 0.15 \times 4)$ = ₦1 516 000

6. $5 = 1.12^n$
$\therefore \log_{1.12} 5 = n$
or, $\frac{\log 5}{\log 1.12} = n$ (Change of base rule)
$\therefore n = 14$
It will take 14 years for the investment to grow.

7. $2 = 1.32^n$
$\therefore \log_{1.32} 2 = n$
or, $\frac{\log 2}{\log 1.32} = n$ (Change of base rule)
$\therefore n = 2.5$
It took two and a half months for food prices to double.

8. Tax payable
$= 21\ 000 + 33\ 000 + 75\ 000 + 95\ 000 + (150\ 000 \times 0.21)$
= ₦255 500

9. $980\ 000 + (2\ 750\ 000 \times 0.05) + (40 \times 18\ 000)$
= ₦1 837 500

10. **a)** $1\ 850\ 000(1.095)^5$ = ₦2 912 341
b) Price for new car: $2\ 275\ 000(1.05)^5$ = ₦2 903 540
Yes, she will be able to buy a new car.

11. **a)** $2\ 500\ 000(1.1)^4$ = ₦3 660 250
b) $(3\ 660\ 250 - 2\ 500\ 000) \div 2\ 500\ 000 \times 100 = 46.41\%$
c) 20%

12. $4\ 500\ 000(1.16)^7 + 3\ 600\ 000(1.16)^3 - 2\ 000\ 000(1.16)^5$
$=12\ 717\ 988.81 + 5\ 619\ 225.60 - 4\ 200\ 683.32$
= ₦14 136 531.09

TOPIC 10

Coordinate geometry

Introduction

Your students are already familiar with the Cartesian plane. They have also worked with the equations of straight lines before, as well as the graphs of straight lines. In this topic, students will use this knowledge as they work with coordinate geometry formally for the first time. Your students will revise what they already know about the Cartesian plane and then derive and use the distance and midpoint formulae. They will work with practical applications of coordinate geometry. Then they will revise how to calculate the gradient of a straight line and use the equation of a straight line to determine the gradient and intercepts of the line, and determine the equation of a straight line using given information. Next, they will find the angle between two intersecting straight lines. Finally, they will apply their knowledge of linear graphs to a variety of real-life situations.

Common difficulties

Common difficulties students experience include:

- When interpreting or plotting points on the Cartesian plane, students must not confuse the x- and y-coordinates.
- Students sometimes use the wrong formula when doing calculations. Refer them to the poster of formulae that you have drawn up and pasted up in your classroom.
- Remind your students to *always* make sure that their calculators are in degree mode when they do trigonometric calculations. If students are unsure how to use the log tables or the trigonometric tables correctly, revise the use of these tables with your class before they start working.
- Students sometimes do the calculations correctly, but then round off their answers incorrectly. Remediate this problem as you become aware of it to discover whether a student has simply been careless, or whether he or she needs additional practice in rounding off numbers.

- When drawing a graph that involves a horizontal and a vertical axis, it is important that the students mark off their scales on the axes evenly and clearly. If they fail to do this, the scale of their graphs will not be correct. Students must label the axes clearly and correctly.
- When calculating distances or lengths, students need only include the units of length in the final answer. If no units of length are given, the students should write a number only, or a number followed by the word "units".
- When solving problems that involve measurement, students sometimes forget to write the correct unit of measurement in the final answer, or they write the wrong units of measurement. Remind your students that, at this level, the correct unit of measurement is as important a part of the answer as the numeric part. Point out that marks are subtracted if the correct unit of measurement is not given and that they can lose marks unnecessarily if they are careless.

Preparation

Prepare the following posters to display in your classroom:
- a poster showing the four quadrants of the Cartesian plane
- a poster showing all the formulae that are used in this topic.

Introducing students to the topic

Briefly revise the following with your class:
- Pythagoras' theorem (Do your students know when to add and when to subtract? Do they realise that the hypotenuse is always the longest side in any right-angled triangle? Do they remember to take the square root at the end of a calculation?)
- the Cartesian plane (Are your students clear about the signs of x and y in the different quadrants?)
- the standard form of the equation of a straight line. (Do your students understand the meanings of m and c in the equation $y = mx + c$?)

Answers

Exercise 10.1 (page 145)

1. **a)** P(−7; 3), Q(−7; −5), R(9; −5)
b) The abscissa of points P and Q is the same: −7.
c) The ordinate of points Q and R is the same: −5.
d) △PQR is a right-angled scalene triangle.
e) S(9; 3)

2. **a)** First quadrant
b) Second quadrant
c) y-axis
d) Third quadrant
e) Second quadrant
f) x-axis
g) First quadrant
h) Fourth quadrant
i) Third quadrant
j) Second quadrant

Exercise 10.2 (page 147)

1. **a)** $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(7 - 3)^2 + (3 - (-2))^2}$
$= \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$ units

b) $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3 - 7)^2 + (-2 - 3)^2}$
$= \sqrt{(-4)^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}$ units

2. The answers to 1(a) and (b) are the same.

3. **a)** $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(12 - 10)^2 + (6 - 4)^2}$
$= \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8}$ units

b) $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[5 - (-5)]^2 + (3 - 11)^2}$
$= \sqrt{10^2 + (-8)^2} = \sqrt{100 + 64} = \sqrt{164}$ units

c) $ST = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1 - 4)^2 + (4 - 9)^2}$
$= \sqrt{(-5)^2 + (-5)^2} = \sqrt{25 + 25} = \sqrt{50}$ units

4. **a)**
$$PQ = 16$$
$$\therefore \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2]} = 16$$
$$\therefore (x_2 - x_1)^2 + (y_2 - y_1)^2 = 16^2$$
$$\therefore (k - (-10))^2 + (-8 - (-8))^2 = 256$$
$$\therefore (k + 10)^2 + 0^2 = 256$$
$$\therefore (k + 10)^2 = 256$$
$$\therefore k + 10 = \pm \sqrt{16}$$

$\therefore k + 10 = 16$ or $k + 10 = -16$
$\therefore k = 6$ or $k = -26$

b) $AB = \sqrt{101}$
$\therefore \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{101}$
$\therefore (x_2 - x_1)^2 + (y_2 - y_1)^2 = 101$
$\therefore (-1 - 9)^2 + (k - 1)^2 = 101$
$\therefore (-10)^2 + (k - 1)^2 = 101$
$\therefore 100 + (k - 1)^2 = 101$
$\therefore (k - 1)^2 = 1$
$\therefore k - 1 = \pm \sqrt{1}$
$\therefore k - 1 = 1$ or $k - 1 = -1$
$\therefore k = 2$ or $k = 0$

c) $EF = \sqrt{13}$
$\therefore \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{13}$
$\therefore (x_2 - x_1)^2 + (y_2 - y_1)^2 = 13$
$\therefore (2k - k)^2 + (8 - 5)^2 = 13$
$\therefore k^2 + 3^2 = 13$
$\therefore k^2 + 9 = 13$
$\therefore k^2 = 4$
$\therefore k = \pm \sqrt{4}$
$\therefore k = 2$ or $k = -2$

Exercise 10.3 (page 149)

1. a) Midpoint $= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{3 + 1}{2}; \frac{-7 - 1}{2}\right) = \left(\frac{4}{2}; \frac{-8}{2}\right)$
$= (2; -4)$

b) Midpoint $= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{0 + 6}{2}; \frac{9 + 3}{2}\right) = \left(\frac{6}{2}; \frac{12}{2}\right) = (3; 6)$

c) Midpoint $= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-5 + 2}{2}; \frac{-10 - 7}{2}\right) = \left(-\frac{3}{2}; -\frac{17}{2}\right)$

d) Midpoint $= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 + 8}{2}; \frac{0 - 2}{2}\right) = \left(\frac{4}{2}; \frac{-2}{2}\right)$
$= (2; -1)$

2. a) Midpoint $= \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-3 + 7}{2}; \frac{5 - 1}{2}\right) = \left(\frac{4}{2}; \frac{4}{2}\right) = (2; 2)$
Coordinates of M: (2; 2)

b) $\frac{x_D + x_F}{2} = x_E$
$\therefore \frac{-3 + x_F}{2} = 7$
$\therefore -3 + x_F = 14$
$\therefore x_F = 14 + 3$
$= 17$

$\frac{y_D + y_F}{2} = y_E$

$\therefore \frac{5 + y_F}{2} = -1$

$\therefore 5 + y_F = -2$

$\therefore y_F = -2 - 5$

$= -7$

Coordinates of F: (17; −7)

c) $\frac{x_C + x_E}{2} = x_D$

$\therefore \frac{x_C + 7}{2} = -3$

$\therefore x_C + 7 = -6$

$\therefore x_C = -6 - 7$

$= -13$

$\frac{y_C + y_E}{2} = x_D$

$\therefore \frac{y_C - 1}{2} = 5$

$\therefore y_C - 1 = 10$

$\therefore y_C = 10 + 1$

$= 11$

Coordinates of CF: (−13; 11)

3. **a)** $\text{Midpoint}_{AE} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-12 + 4}{2}; \frac{10 - 14}{2}\right) = \left(\frac{-8}{2}; \frac{-4}{2}\right)$

$= (-4; -2)$

Coordinates of C: (−4; −2)

b) $\text{Midpoint}_{AC} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-12 - 4}{2}; \frac{10 - 2}{2}\right) = \left(\frac{-16}{2}; \frac{8}{2}\right)$

$= (-8; 4)$

Coordinates of B: (−8; 4)

c) $\text{Midpoint}_{CE} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 + 4}{2}; \frac{-2 - 14}{2}\right) = \left(\frac{0}{2}; \frac{-16}{2}\right)$

$= (0; -8)$

Coordinates of D: (0; −8)

4. **a)** Q(3; 1)

b) $\text{Midpoint}_{PR} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-1 + 2}{2}; \frac{3 - 1}{2}\right) = \left(\frac{1}{2}; 1\right)$

$\text{Midpoint}_{SQ} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-2 + 3}{2}; \frac{1 + 1}{2}\right) = \left(\frac{1}{2}; 1\right)$

The diagonals have the same midpoint, so they bisect one another.

Exercise10.4 (page 150)

1. a) **i)** B(8; 1) **ii)** C(1; 2) **iii)** D(8; 6)
iv) F(0; 8) **v)** I(3; 7) **vi)** M(5; 4)
vii) P(10; 8) **viii)** S(9; 1)

b) **i)** FP(5; 8) **ii)** DP(9; 7)
iii) BD(8; 3.5) **iv)** BS(8.5; 1)

c) **i)** BD = 5 units = 500 m (By inspection)

ii) $DP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(10 - 8)^2 + (8 - 6)^2}$
$= \sqrt{2^2 + 2^2}$
$= \sqrt{4 + 4}$
$= \sqrt{8}$ units
$= 283$ m

iii) $FM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(5 - 0)^2 + (4 - 8)^2}$
$= \sqrt{5^2 + (-4)^2}$
$= \sqrt{25 + 16}$
$= \sqrt{41}$ units
$= 640$ m

v) $CI = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(3 - 1)^2 + (7 - 2)^2}$
$= \sqrt{2^2 + 5^2}$
$= \sqrt{4 + 25}$
$= \sqrt{29}$ units
$= 539$ m

d) Using the distance formula:
$IM = \sqrt{13}$ units
$DM = \sqrt{13}$ units
$DI = \sqrt{26}$ units
$DI^2 = IM^2 + DM^2$ and IM = DM
So, $\triangle$DIM is a right-angled isosceles triangle.

e) $CS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(9 - 1)^2 + (1 - 2)^2}$
$= \sqrt{8^2 + (-1)^2} = \sqrt{64 + 1} = \sqrt{65}$ units = 806 m

$IS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(9 - 3)^2 + (1 - 7)^2}$

$= \sqrt{6^2 + (-6)^2} = \sqrt{36 + 36} = \sqrt{72}$ units = 849 m

Chinasa lives closer to the school.

2. **a)** Let $(x_1; y_1) = (0; 0)$ and $(x_2; y_2) = (105; 68)$.

Length of diagonal $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(68 - 0)^2 + (105 - 0)^2} = \sqrt{68^2 + 105^2}$

$= \sqrt{4\,624 + 11\,025} = \sqrt{15\,649} = 125$ m

b) $P = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{105 + 0}{2}; \frac{68 + 0}{2}\right) = \left(\frac{105}{2}; \frac{68}{2}\right) = (52.5; 34)$

c) **i)**

$$\frac{x_Q + x_R}{2} = x_P$$
$$\therefore \frac{24 + x_R}{2} = 52.5$$
$$\therefore 24 + x_R = 105$$
$$\therefore x_R = 105 - 24$$
$$= 81$$
$$\frac{y_Q + y_R}{2} = y_P$$
$$\therefore \frac{10 + y_R}{2} = 34$$
$$\therefore 10 + y_R = 68$$
$$\therefore y_R = 68 - 10$$
$$= 58$$

Coordinates of R: (81; 58)

ii) Let $(x_1; y_1) = (24; 10)$ and $(x_2; y_2) = (81; 58)$.

$QR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

$= \sqrt{(58 - 10)^2 + (81 - 24)^2}$

$= \sqrt{48^2 + 57^2}$

$= \sqrt{2\,304 + 3\,249}$

$= \sqrt{5\,553}$

$= 75$ m

d) Let $(x_1; y_1) = (70; 15)$ and $(x_2; y_2) = (106; 38)$.

$ST = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(106 - 70)^2 + (38 - 15)^2}$

$= \sqrt{36^2 + 23^2}$

$= \sqrt{1\,296 + 529}$

$= \sqrt{1\,825}$

$= 43$ m

3. **a)** **i)** Yola: (52; 31) **ii)** Kano: (33; 46)
iii) Enugu: (27; 16) **iv)** Maiduguri: (55; 46)
v) Abuja: (27; 30) **vi)** Lagos: (6; 17)

b) 10 units ≈ 220 km; so, 1 unit ≈ 22 km

i) Let $(x_1; y_1) = (27; 30)$ and $(x_2; y_2) = (26; 37)$.
Distance between Kauna and Abuja
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(26 - 27)^2 + (37 - 30)^2}$
$= \sqrt{(-1)^2 + (7)^2}$
$= \sqrt{1 + 49}$
$= \sqrt{50}$ units
= 156 km

ii) Let $(x_1; y_1) = (33; 46)$ and $(x_2; y_2) = (27; 16)$.
Distance between Kano and Enugu
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(27 - 33)^2 + (16 - 46)^2}$
$= \sqrt{(-6)^2 + (-30)^2}$
$= \sqrt{36 + 900}$
$= \sqrt{936}$
= 673 km

iii) Let $(x_1; y_1) = (6; 17)$ and $(x_2; y_2) = (52; 31)$.
Distance between Lagos and Yola
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(52 - 6)^2 + (31 - 17)^2}$
$= \sqrt{46^2 + 14^2}$
$= \sqrt{2\,116 + 196}$
$= \sqrt{2\,312}$
= 1 058 km

iv) Let $(x_1; y_1) = (55; 46)$ and $(x_2; y_2) = (33; 46)$.
Distance between Maiduguri and Kano
$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(33 - 55)^2 + (46 - 46)^2}$
$= \sqrt{(-22)^2 + (0)^2}$
$= \sqrt{484}$ units
= 484 km

Exercise 10.5 (page 153)

1. a) $m_{AE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 3}{1 - (-4)} = \frac{1}{5}$

   b) $m_{BE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{-4 - (-2)} = \frac{6}{-2} = -3$

   c) $m_{AC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{1 - 1} = \frac{4}{0}$; undefined

   d) $m_{BD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - (-3)}{4 - (-2)} = \frac{0}{6} = 0$

   e) $m_{DE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{-4 - 4} = \frac{6}{-8} = -\frac{3}{4}$

   f) $m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-3)}{1 - (-2)} = \frac{3}{3} = 1$

2. a) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 3}{-3 - (-8)} = \frac{5}{5} = 1$

   b) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-1)}{5 - 6} = \frac{5}{-1} = -5$

   c) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{0 - 9} = \frac{3}{-9} = -\frac{1}{3}$

   d) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 2}{-9 - (-7)} = \frac{4}{-2} = -2$

3. a) $m_{DE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 7}{6 - 5} = \frac{-10}{1} = -10$

   b) $m_{EF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-3)}{-4 - 6} = \frac{2}{-10} = -\frac{1}{5}$

   c) $m_{FG} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-1)}{-2 - (-4)} = \frac{8}{2} = 4$

   d) $m_{DG} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 7}{-2 - 5} = \frac{0}{-7} = 0$

   e) $m_{DF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 7}{-4 - 5} = \frac{-8}{-9} = \frac{8}{9}$

   f) $m_{EG} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-3)}{-2 - 6} = \frac{10}{-8} = -\frac{5}{4}$

Exercise 10.6 (page 155)

1. a) This equation is in standard form ($y = mx + c$).

   i) Gradient: $m = -4$

   ii) y-intercept: $c = -6$

   iii) To find the x-intercept, set $y = 0$:

   $$0 = -4x - 6$$
   $$\therefore 4x = -6$$
   $$\therefore x = -\frac{6}{4}$$
   $$\therefore x = -\frac{3}{2}$$

   So, the x-intercept is $-\frac{3}{2}$.

   b) This equation is in standard form ($y = mx + c$).

   i) Gradient: $m = -1$

   ii) y-intercept: $c = 0$

   iii) To find the x-intercept, set $y = 0$:

$0 = -x$

$\therefore x = 0$

So, the x-intercept = 0.

c) This equation is in standard form ($y = mx + c$).

i) Gradient: $m = 0.3$

ii) y-intercept: $c = -7.2$

iii) To find the x-intercept, set $y = 0$:

$0 = 0.3x - 7.2$

$\therefore -0.3x = -7.2$

$\therefore 0.3x = 7.2$

$\therefore x = \frac{7.2}{0.3}$

$\therefore x = 24$

So, the x-intercept = 24.

d) This equation is in standard form ($y = mx + c$).

i) Gradient: $m = \frac{3}{4}$

ii) y-intercept: $c = 1$

iii) To find the x-intercept, set $y = 0$:

$0 = \frac{3}{4}x + 1$

$\therefore -\frac{3}{4}x = 1$

$\therefore x = -\frac{4}{3}$

So, the x-intercept = $-\frac{4}{3}$.

2. **a)** First write the equation in standard form.

$-5y = -3x + 30$

$\therefore 5y = 3x - 30$

$\therefore y = \frac{3}{5}x - 6$

i) Gradient: $m = \frac{3}{5}$

ii) y-intercept: $c = -6$

iii) To find the x-intercept, set $y = 0$:

$3x - 5(0) = 30$

$\therefore 3x = 30$

$\therefore x = 10$

So, the x-intercept = 10.

b) First write the equation in standard form.

$y = -2x + 6$

i) Gradient: $m = -2$

ii) y-intercept: $c = 6$

iii) To find the x-intercept, set $y = 0$:

$0 = -2x + 6$

$\therefore 2x = 6$

$\therefore x = 3$

So, the x-intercept = 3.

c) First write the equation in standard form.

$1.5y - 6x = -7.5$

$\therefore 1.5y = 6x - 7.5$

$\therefore y = 4x - 5$

i) Gradient: $m = 4$

ii) y-intercept: $c = -5$

iii) To find the x-intercept, set $y = 0$:

$0 = 4x - 5$

$\therefore -4x = 5$

$\therefore x = -\frac{5}{4}$

So, the x-intercept $= -\frac{5}{4}$.

d) First write the equation in standard form.

$3y + 9 = 2x$

$\therefore 3y = 2x - 9$

$\therefore y = \frac{2}{3}x - 3$

i) Gradient: $m = \frac{2}{3}$

ii) y-intercept: $c = -3$

iii) To find the x-intercept, set $y = 0$:

$3(0) + 9 = 2x$

$\therefore 2x = 9$

$\therefore x = \frac{9}{2}$

So, the x-intercept $= \frac{9}{2}$.

e) First write the equation in standard form.

$2(y + 4) = 5x$

$\therefore 2y + 8 = 5x$

$\therefore 2y = 5x - 8$

$\therefore y = \frac{5}{2}x - 4$

i) Gradient: $m = \frac{5}{2}$

ii) y-intercept: $c = -4$

iii) To find the x-intercept, set $y = 0$:

$0 = 5x - 8$

$\therefore -5x = -8$

$\therefore x = \frac{-8}{-5}$

$\therefore x = \frac{8}{5}$

So, the x-intercept $= \frac{8}{5}$.

f) First write the equation in standard form.

$7 = 4x - y$

$\therefore y = 4x - 7$

i) Gradient: $m = 4$

ii) y-intercept: $c = -7$

iii) To find the x-intercept, set $y = 0$:

$$0 = 4x - 7$$
$$\therefore -4x = -7$$
$$\therefore x = \tfrac{-7}{-4}$$
$$\therefore x = \tfrac{7}{4}$$

So, the x-intercept $= \tfrac{7}{4}$.

Exercise 10.7 (page 158)

1. a) The gradient and the y-intercept are known, so use the equation, $y = mx + c$.

$m = -1$ and $c = \tfrac{2}{3}$

$\therefore y = -x + \tfrac{2}{3}$

b) The gradient and one point are known, so use the equation, $y - y_1 = m(x - x_1)$.

$m = -4$ and $(x_1; y_1) = (2; 0)$

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 0 = -4(x - 2)$$
$$\therefore y = -4x + 8$$

c) The gradient and one point are known, so use the equation, $y - y_1 = m(x - x_1)$.

$m = \tfrac{3}{5}$ and $(x_1; y_1) = (5; 5)$

$$y - y_1 = m(x - x_1)$$
$$\therefore y - 5 = \tfrac{3}{5}(x - 5)$$
$$\therefore y - 5 = \tfrac{3}{5}x - 3$$
$$\therefore y = \tfrac{3}{5}x + 2$$

d) Two points are known, so use the equation,

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

$(x_1; y_1) = (1; -9)$ and $(x_2; y_2) = (-1; 3)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$\therefore y - (-9) = \frac{3 - (-9)}{-1 - 1}(x - 1)$$
$$\therefore y + 9 = \frac{12}{-2}(x - 1)$$
$$\therefore y = -6(x - 1) - 9$$
$$\therefore y = -6x + 6 - 9$$
$$\therefore y = -6x - 3$$

e) Two points are known, one is the y-intercept; so, we can use different methods. In this solution, we use method 1.

Two points are known, so use the equation,
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
$(x_1; y_1) = (5; 0)$ and $(x_2; y_2) = (0; -10)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$\therefore y - 0 = \frac{-10 - 0}{0 - 5}(x - 5)$$
$$\therefore y = 2(x - 5)$$
$$\therefore y = 2x - 10$$

f) The gradient and one point are known, so use the equation: $y - y_1 = m(x - x_1)$.
$m = 1$ and $(x_1; y_1) = (2; -2)$

$$y - y_1 = m(x - x_1)$$
$$\therefore y - (-2) = 1(x - 2)$$
$$\therefore y + 2 = x - 2$$
$$\therefore y = x - 4$$

g) The line is perpendicular to a line with a gradient of 2, so the gradient of this line is $-\frac{1}{2}$ (as $2 \times -\frac{1}{2} = -1$).
The gradient and the y-intercept are known, so use the equation, $y = mx + c$.
$m = -\frac{1}{2}$ and $c = 7$
$\therefore y = -\frac{1}{2}x + 7$

h) Two points are known, one is the y-intercept; so we can use different methods. In this solution, we use method 2.
First calculate the gradient:
$(x_1; y_1) = (-10; 0)$ and $(x_2; y_2) = (0; 12)$
$\therefore m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12 - 0}{0 - (-10)} = \frac{12}{10} = \frac{6}{5}$
We now have the gradient and the y-intercept, so we use the equation, $y = mx + c$.
$m = \frac{6}{5}$ and $c = 12$
$\therefore y = \frac{6}{5}x + 12$

i) Two points are known, so use the equation,
$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.
$(x_1; y_1) = (2; -1)$ and $(x_2; y_2) = (6; 5)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$\therefore y - (-1) = \frac{5 - (-1)}{6 - 2}(x - 2)$$
$$\therefore y + 1 = \frac{6}{4}(x - 2)$$
$$\therefore y = \frac{3}{2}(x - 2) - 1$$

$\therefore y = \frac{3}{2}x - 3 - 1$

$\therefore y = \frac{3}{2}x - 4$

j) The line is parallel to a line with a gradient of $-\frac{1}{10}$; so, the gradient of this line is also $-\frac{1}{10}$.
The gradient and one point are known, so use the equation, $y - y_1 = m(x - x_1)$.

$m = -\frac{1}{10}$ and $(x_1; y_1) = (5; 0)$

$y - y_1 = m(x - x_1)$

$\therefore y - 0 = -\frac{1}{10}(x - 5)$

$\therefore y = -\frac{1}{10}x + \frac{1}{2}$

2. **a)** Two points are known, one is the y-intercept; so we can use different methods. In this solution, we use method 1.
Two points are known, so we use the equation,

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$(x_1; y_1) = (0; -1)$ and $(x_2; y_2) = (2; 1)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$\therefore y - (-1) = \frac{1 - (-1)}{2 - 0}(x - 0)$

$\therefore y + 1 = \frac{2}{2}x$

$\therefore y = x - 1$

b) Two points are known; one is the y-intercept; so we can use different methods. In this solution, we use method 2.
First calculate the gradient:

$(x_1; y_1) = (4; 0)$ and $(x_2; y_2) = (0; 3)$

$\therefore m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{3 - 0}{0 - 4}$

$= \frac{3}{-4}$

$= -\frac{3}{4}$

The gradient and the y-intercept are known; so, use the equation, $y = mx + c$.

$m = -\frac{3}{4}$ and $c = 3$

$\therefore y = -\frac{3}{4}x + 3$

c) Two points are known, one is the y-intercept, so we can use different methods. In this solution, we use method 2.

First calculate the gradient:

$(x_1; y_1) = (0; 0)$ and $(x_2; y_2) = (4; -8)$

$\therefore m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-8 - 0}{4 - 0}$

$= \frac{-8}{4}$

$= -2$

The gradient and the y-intercept are known, so we use the equation, $y = mx + c$.

$m = -2$ and $c = 0$

$\therefore y = -2x + 0$

$\therefore y = -2x$

d) Two points are known, so use the equation:

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

$(x_1; y_1) = (-3; -2)$ and $(x_2; y_2) = (2; 3)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$\therefore y - (-2) = \frac{3 - (-2)}{2 - (-3)}(x - (-3))$

$\therefore y + 2 = \frac{5}{5}(x + 3)$

$\therefore y = x + 3 - 2$

$\therefore y = x + 1$

Exercise 10.8 (page 160)

1. **a)** $m_{PQ} = 1$

$\therefore \tan\theta = 1$

$\therefore \theta = \tan^{-1} 1$

$= 45°$

b) $m_{PQ} = -1$

$\therefore \tan\theta = -1$

$\therefore \theta = 180° - \tan^{-1} 1$

$= 180° - 45°$

$= 135°$

c) $m_{PQ} = \frac{1}{4}$

$\therefore \tan\theta = \frac{1}{4}$

$\therefore \theta = \tan^{-1} \frac{1}{4}$

$= 14.0°$

d) $m_{PQ} = -\frac{1}{4}$

$\therefore \tan\theta = -\frac{1}{4}$

$\therefore \theta = 180° - \tan^{-1}\frac{1}{4}$

$= 180° - 14.0°$

$= 166.0°$

e) $m_{PQ} = \frac{4}{11}$

$\therefore \tan\theta = \frac{4}{11}$

$\therefore \theta = \tan^{-1}\frac{4}{11}$

$= 20.0°$

f) $m_{PQ} = -\frac{4}{11}$

$\therefore \tan\theta = -\frac{4}{11}$

$\therefore \theta = 180° - \tan^{-1}\frac{4}{11}$

$= 180° - 20.0°$

$= 160.0°$

g) $m_{PQ} = 2$

$\therefore \tan\theta = 2$

$\therefore \theta = \tan^{-1} 2$

$= 63.4°$

h) $m_{PQ} = -2$

$\therefore \tan\theta = -2$

$\therefore \theta = 180° - \tan^{-1} 2$

$= 180° - 63.4°$

$= 116.6°$

2. a) $m = \frac{3}{2}$

$\therefore \tan\theta = \frac{3}{2}$

$\therefore \theta = \tan^{-1}\frac{3}{2}$

$= 56.3°$

b) $m = -\frac{2}{5}$

$\therefore \tan\theta = -\frac{2}{5}$

$\therefore \theta = 180° - \tan^{-1}\frac{2}{5}$

$= 180° - 21.8°$

$= 158.2°$

Exercise 10.9 (page 161)

1. $m_{AB} = -1$

$\therefore \tan\alpha = -1$

$\therefore \alpha = 180° - \tan^{-1} 1$

$= 180° - 45°$

$= 135°$

$m_{CD} = 2$
$\therefore \tan\beta = 2$
$\therefore \beta = \tan^{-1} 2$
$= 63.4°$
$\theta = (180° - \alpha) + \beta$ (Exterior angle of triangle)
$= 45° + 63.4°$
$= 108.4°$

2. $m_{AB} = \frac{1}{2}$
$\therefore \tan\alpha = \frac{1}{2}$
$\therefore \alpha = \tan^{-1} \frac{1}{2}$
$= 26.6°$
$m_{CD} = -4$
$\therefore \tan\beta = -4$
$\therefore \beta = 180° - \tan^{-1} 4$
$= 180° - 76.0°$
$= 104.0°$
$\theta = \beta - \alpha$ (Exterior angle of triangle)
$= 104.0° - 26.6°$
$= 77.4°$

3. $m_{AB} = -\frac{3}{5}$
$\therefore \tan\alpha = -\frac{3}{5}$
$\therefore \alpha = 180° - \tan^{-1} \frac{3}{5}$
$= 180° - 31.0°$
$= 149.0°$
$m_{CD} = \frac{2}{3}$
$\therefore \tan\beta = \frac{2}{3}$
$\therefore \beta = \tan^{-1} \frac{2}{3}$
$= 33.7°$
$\theta = (180° - \alpha) + \beta$ (Exterior angle of triangle)
$= 31.0° + 33.7°$
$= 64.7°$

4. $m_{AB} = 6$
$\therefore \tan\alpha = 6$
$\therefore \alpha = \tan^{-1} 6$
$= 80.5°$
$m_{CD} = 1$
$\therefore \tan\beta = 1$
$\therefore \beta = 45°$

$\theta = (180° - \alpha) + \beta$ (Exterior angle of triangle)
$= (180° - 80.5°) + 45°$
$= 144.5°$

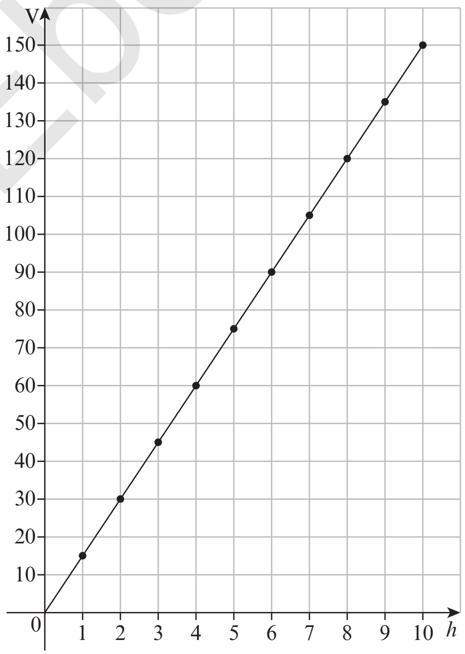
5. $m_{AB} = -\frac{8}{5}$
$\therefore \tan\alpha = -\frac{8}{5}$
$\therefore \alpha = 180° - \tan^{-1}\frac{8}{5}$
$= 180° - 58.0°$
$= 122.0°$
$m_{CD} = -\frac{1}{2}$
$\therefore \tan\beta = -\frac{1}{2}$
$\therefore \beta = 180° - \tan^{-1}\frac{1}{2}$
$= 180° - 26.6°$
$= 153.4°$
$\theta = \alpha + (180° - \beta)$ (Exterior angle of triangle)
$= 122° + 26.6°$
$= 148.6°$

Exercise 10.10 (page 164)

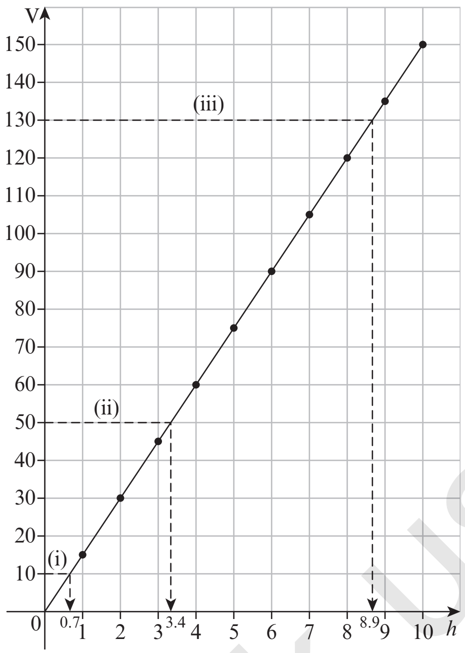
1. **a)** $V = (5)(3)h = 15h$

b) It is a linear relationship because it is of the form $y = mx$.

c)



d)



Below are the accurate answers.

i) 0.7 m

ii) 3.3 cm

iii) 8.7 mm

e) $V = lbh$

$\therefore h = \frac{V}{15}$

i) $h = \frac{10}{15} = 0.7$ m

ii) $h = \frac{50}{15} = 3.3$ cm

iii) $h = \frac{130}{15} = 8.7$ mm

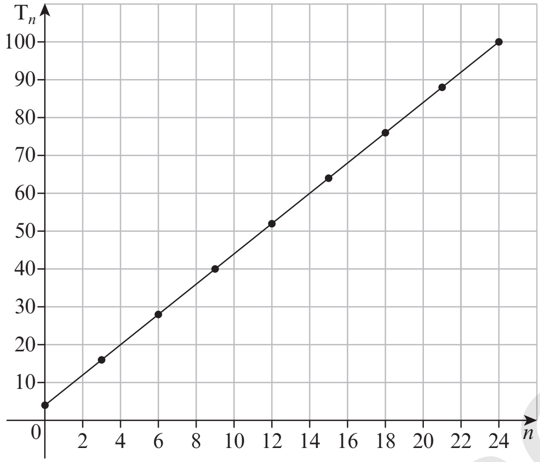
2. a) $T_n = 8 + (n - 1)(4) = 8 + 4n - 4 = 4n + 4$

b) It is a linear relationship because it is of the form $y = mx + c$.

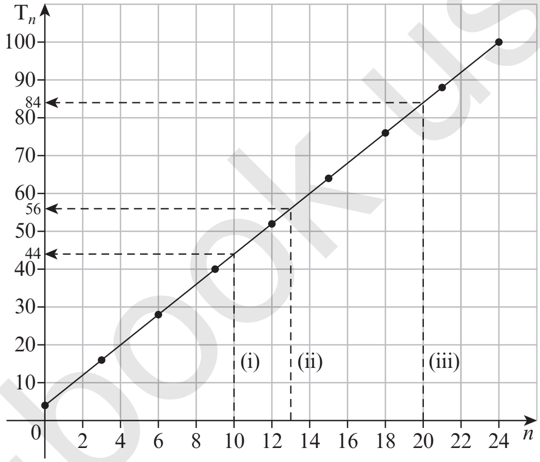
c)

n	0	3	6	9	12	15	18	21	24
T_n	4	16	28	40	52	64	76	88	100

d)



e)



Below are the accurate answers.

i) 44

ii) 56

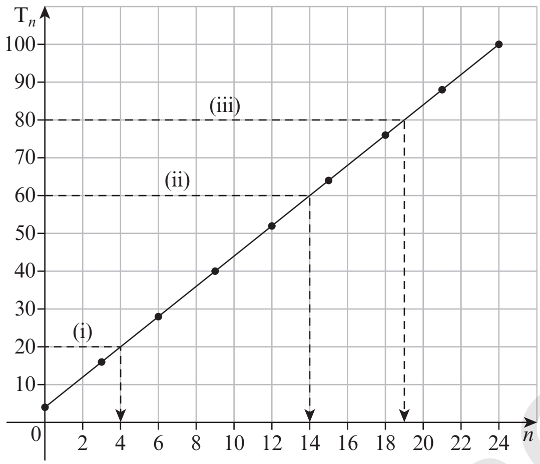
iii) 84

f) **i)** $T_n = 4n + 4; \therefore T_{10} = 4(10) + 4 = 44$

ii) $T_n = 4n + 4; \therefore T_{13} = 4(13) + 4 = 56$

iii) $T_n = 4n + 4; \therefore T_{20} = 4(20) + 4 = 84$

g)



Below are the accurate answers.

i) T_4

ii) T_{14}

iii) T_{19}

Note that the term number must always be a positive integer. It is not possible to have a fractional term number or a negative term number.

h) $T_n = 4n + 4$

$\therefore 4n = T_n - 4$

$\therefore n = \frac{T_n}{4} - 1$

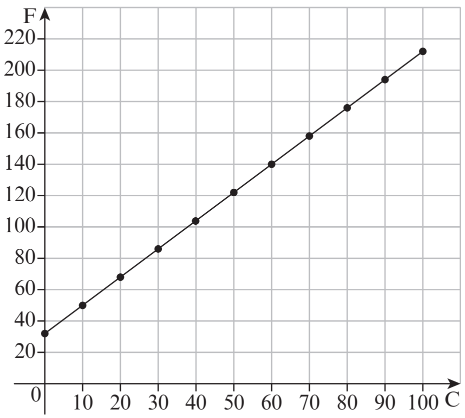
i) **i)** $n = \frac{T_n}{4} - 1 = \frac{20}{4} - 1 = 5 - 1 = 4$

ii) $n = \frac{T_n}{4} - 1 = \frac{60}{4} - 1 = 15 - 1 = 14$

iii) $n = \frac{T_n}{4} - 1 = \frac{80}{4} - 1 = 20 - 1 = 19$

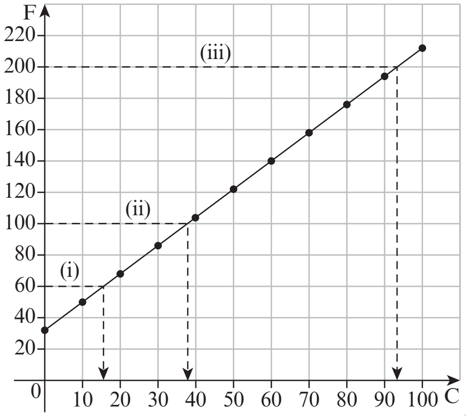
3. **a)** It is a linear relationship because it is of the form $y = mx + c$.

b)



c) 32 °F **d)** 212 °F

e)



Below are the accurate answers.

i) 16 °C

ii) 38 °C

iii) 93 °C

f) $F = \frac{9}{5}C + 32$

$\therefore \frac{9}{5}C = F - 32$

$\therefore 9C = 5F - 160$

$\therefore C = \frac{5F - 160}{9}$

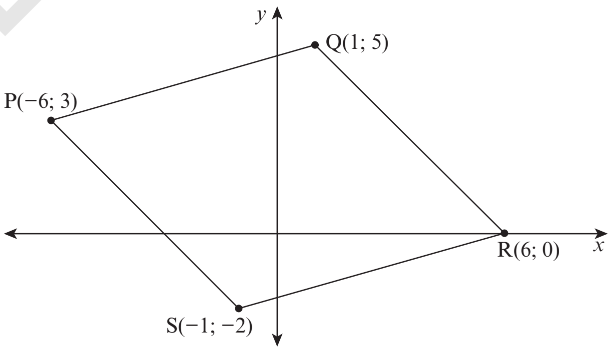
g) **i)** $C = \frac{5F - 160}{9} = \frac{5(60) - 160}{9} = 16$ °C

ii) $C = \frac{5F - 160}{9} = \frac{5(100) - 160}{9} = 38$ °C

iii) $C = \frac{5F - 160}{9} = \frac{5(200) - 160}{9} = 93$ °C

Assess your progress (page 165)

1. a)



b) **i)** $m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{1 - (-6)} = \frac{2}{7}$

$m_{RS} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-2)}{6 - (-1)} = \frac{2}{7}$

$m_{PQ} = m_{RS}$; so, PQ || RS

ii) $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[1 - (-6)]^2 + (5 - 3)^2}$
$= \sqrt{7^2 + 2^2}$
$= \sqrt{49 + 4}$
$= \sqrt{53}$ units

$RS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[6 - (-1)]^2 + [0 - (-2)]^2}$
$= \sqrt{7^2 + 2^2}$
$= \sqrt{49 + 4}$
$= \sqrt{53}$ units

$\therefore PQ = RS$

iii) $\text{Midpoint}_{PR} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-6 + 6}{2}; \frac{3 + 0}{2}\right)$
$= \left(\frac{0}{2}; \frac{3}{2}\right) = \left(0; \frac{3}{2}\right)$

$\text{Midpoint}_{SQ} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-1 + 1}{2}; \frac{-2 + 5}{2}\right)$
$= \left(\frac{0}{2}; \frac{3}{2}\right) = \left(0; \frac{3}{2}\right)$

The diagonals have the same midpoint, so they bisect each another.

c) If PQRS is a rhombus, its diagonals are perpendicular. Or, if PQRS is a rhombus, then PQ = QR.
It is easier to calculate the gradients of the diagonals.

$m_{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{-6 - 6} = \frac{3}{-12} = -\frac{1}{4}$

$m_{QS} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-2)}{1 - (-1)} = \frac{7}{2}$

$m_{PR} \times m_{QS} = -\frac{1}{4} \times \frac{7}{2} = -\frac{7}{8} \neq -1$

So, the gradients of the diagonals are not perpendicular and, therefore, PQRS is not a rhombus.

2. **a)** This equation is in standard form $y = mx + c$.

i) Gradient: $m = \frac{5}{2}$

ii) y-intercept: $c = 3$

iii) To find the x-intercept, set $y = 0$:

$0 = \frac{5}{2}x + 3$
$\therefore -\frac{5}{2}x = 3$
$\therefore -5x = 6$
$\therefore x = -\frac{6}{5}$

So, the x-intercept is $-\frac{6}{5}$.

b) First write the equation in standard form.

$6y - 3x + 18 = 0$

$\therefore 6y = 3x - 18$

$\therefore y = \frac{3}{6}x - 3$

$\therefore y = \frac{1}{2}x - 3$

i) Gradient: $m = \frac{1}{2}$

ii) y-intercept: $c = -3$

iii) To find the x-intercept, set $y = 0$:

$\frac{1}{2}x = 3$

$\therefore x = 6$

So, the x-intercept = 6.

3. This line is perpendicular to a line with a gradient of −4, so the gradient of this line is $\frac{1}{4}$ (as $-4 \times \frac{1}{2} = -1$).

The gradient and one point are known, so use the equation: $y - y_1 = m(x - x_1)$.

$m = \frac{1}{4}$ and $(x_1; y_1) = (8; 1)$

$y - y_1 = m(x - x_1)$

$\therefore y - 1 = \frac{1}{4}(x - 8)$

$\therefore y - 1 = \frac{1}{4}x - 2$

$\therefore y = \frac{1}{4}x - 1$

4. $m_{AB} = \frac{3}{4}$

$\therefore \tan \alpha = \frac{3}{4}$

$\therefore \alpha = \tan^{-1} \frac{3}{4}$

$= 36.9°$

$m_{CD} = -\frac{3}{5}$

$\therefore \tan \beta = -\frac{3}{5}$

$\therefore \beta = 180° - \tan^{-1} \frac{3}{5}$

$= 180° - 31.0°$

$= 149.0°$

$\theta = \alpha + (180° - \beta)$ (Exterior angle of triangle)

$= 36.9° + 31.0°$

$= 67.9°$

5. a) Total surface area $= 2(lh + lb + bh)$

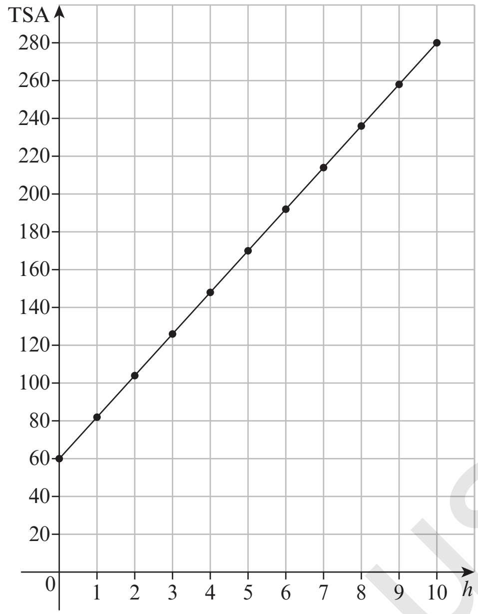
$= 2(6h + 6 \times 5 + 5h)$

$= 2(11h + 30)$

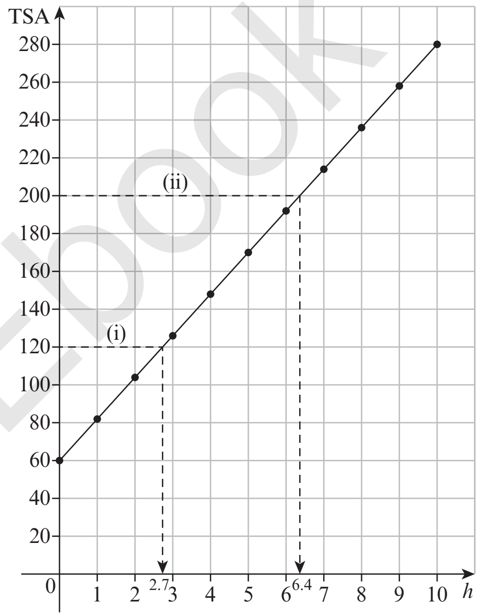
$= 22h + 60$

b) It is a linear relationship because it is of the form $y = mx + c$.

c)



d)



Below are the accurate answers.

i) 2.7 cm

ii) 6.4 mm

TOPIC 11 Differentiation of algebraic functions

Introduction

In this topic, you will introduce your students to calculus for the first time. Your students will learn about limits of functions and the differentiation of simple algebraic functions. They will work with the sum rule, the difference rule, the product rule and the quotient rule. We use these rules to calculate the derivatives of composite functions. Your students will apply their knowledge of differentiation when they solve problems that are set in a variety of real-life situations and that involve maxima and minima, as well as velocity, acceleration and rate of change.

Common difficulties

Students often experience the following problems:

- One of the most common problems that students find when doing differentiation is that they are not sure exactly how to use each type of notation. For example, when differentiating a function from first principles, they are not sure when they must write $\lim_{h \to 0}$ and at what stage they must no longer write it. Tell students that they must write $\lim_{h \to 0}$ in every step in which h appears in the denominator. When they have cancelled the h out, they can stop including $\lim_{h \to 0}$. This notation falls away in the step in which they substitute 0 in the place of h throughout.
- When differentiating root forms, your students will work with common fractions. In particular, they need to be able to add, subtract and multiply with common fractions. Even at this level, a few students will not be confident about calculations with fractions. Remediate problems as they arise. If many of your students struggle with fractions, work through a few examples on the board with the whole class.
- As always, advise your class to be very careful when multiplying different signs. It is easy to make mistakes when working with different signs. Such errors will

cost your students unnecessary marks in tests and examinations.

Preparation for this topic

Prepare the following posters to display in your classroom:

- examples of differentiating from first principles
- the different rules used in differentiation
- the different formulae that are used in this topic
- the basic laws of exponents, with examples
- the different notations that we use when doing differentiation and examples of how we use each one.

Introducing students to the topic

Tell your students that they will learn about calculus for the first time this year. Calculus is a branch of mathematics that is used widely in many fields of science, engineering and economics.

If any of your students are considering future studies that will involve Mathematics, they will probably learn a lot more about calculus.

The main focus of this topic is differentiation, but in the next topic, students will also learn about integration.

Briefly revise the gradient formula with your class. Point to the posters that you have prepared and displayed in your classroom and tell your class that these posters summarise the key information that they will need as they work through this topic.

Answers

Exercise 11.1 (page 170)

1. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2)$
$= 2 + 2 = 4$

2. $\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2} \frac{(x + 2)(x - 2)}{x + 2} = \lim_{x \to -2} (x - 2)$
$= -2 - 2 = -4$

3. $\lim_{x \to -9} \frac{81 - x^2}{2x + 18} = \lim_{x \to -9} \frac{(9 + x)(9 - x)}{2(x + 9)} = \lim_{x \to -9} \frac{9 - x}{2}$
$= \frac{9 - (-9)}{2} = \frac{18}{2} = 9$

4. $\lim_{x \to 0} \frac{x^3 + 6x}{x} = \lim_{x \to 0} \frac{x(x^2 + 6)}{x} = \lim_{x \to 0} (x^2 + 6)$
$= 0^2 + 6 = 6$

5. $\lim_{x \to 3} \frac{x^2 - 5x + 6}{2x - 6} = \lim_{x \to 3} \frac{(x - 3)(x - 2)}{2(x - 3)} = \lim_{x \to 3} \frac{x - 2}{2}$
$= \frac{3 - 2}{2} = \frac{1}{2}$

6. $\lim_{x \to 0} \frac{x^2 + 5x}{x^2 - 7x} = \lim_{x \to 0} \frac{x(x + 5)}{x(x - 7)} = \lim_{x \to 0} \frac{x + 5}{x - 7}$
$= \frac{0 + 5}{0 - 7} = -\frac{5}{7}$

Exercise 11.2 (page 173)

1. **a)** $f(x) = 4x$

$\therefore f(x + h) = 4(x + h) = 4x + 4h$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4x + 4h - 4x}{h}$

$= \lim_{h \to 0} \frac{4h}{h} = \lim_{h \to 0} 4 = 4$

b) $f(x) = x^2$

$\therefore f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x + 0$

$= 2x$

c) $f(x) = -6x^2$

$\therefore f(x + h) = -6(x + h)^2 = -6(x^2 + 2xh + h^2)$

$= -6x^2 - 12xh - 6h^2$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-6x^2 - 12xh - 6h^2 - (-6x^2)}{h}$

$= \lim_{h \to 0} \frac{-6x^2 - 12xh - 6h^2 + 6x^2}{h}$

$= \lim_{h \to 0} \frac{-12xh - 6h^2}{h}$

$= \lim_{h \to 0} (-12x - 6h)$

$= -12x - 6(0) = -12x$

d) $f(x) = 2x^2 + 5x$

$\therefore f(x + h) = 2(x + h)^2 + 5(x + h)$

$= 2(x^2 + 2xh + h^2) + 5x + 5h$

$= 2x^2 + 4xh + 2h^2 + 5x + 5h$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 5x + 5h - (2x^2 + 5x)}{h}$

$= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 5x + 5h - 2x^2 - 5x}{h}$

$= \lim_{h \to 0} \frac{4xh + 2h^2 + 5h}{h}$
$= \lim_{h \to 0} (4x + 2h + 5)$
$= 4x + 2(0) + 5$
$= 4x + 5$

e) $f(x) = -x^3 + x$
$\therefore f(x + h) = -(x + h)^3 + (x + h)$
$= -(x + h)(x + h)^2 + x + h$
$= -(x + h)(x^2 + 2xh + h^2) + x + h$
$= -(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) + x + h$
$= -x^3 - 2x^2h - xh^2 - x^2h - 2xh^2 - h^3 + x + h$
$= -x^3 - 3x^2h - 3xh^2 - h^3 + x + h$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{-x^3 - 3x^2h - 3xh^2 - h^3 + x + h - (-x^3 + x)}{h}$
$= \lim_{h \to 0} \frac{-x^3 - 3x^2h - 3xh^2 - h^3 + x + h + x^3 - x}{h}$
$= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3 + h}{h}$
$= \lim_{h \to 0} (-3x^2 - 3xh - h^2 + 1)$
$= -3x^2 - 3x(0) - 0^2 + 1$
$= -3x^2 + 1$

f) $f(x) = 2x^3 - 8$
$\therefore f(x + h) = 2(x + h)^3 - 8$
$= 2(x + h)(x + h)^2 - 8$
$= 2(x + h)(x^2 + 2xh + h^2) - 8$
$= 2(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) - 8$
$= 2x^3 + 4x^2h + 2xh^2 + 2x^2h + 4xh^2 + 2h^3 - 8$
$= 2x^3 + 6x^2h + 6xh^2 + 2h^3 - 8$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 8 - (2x^3 - 8)}{h}$
$= \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 8 - 2x^3 + 8}{h}$
$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h}$
$= \lim_{h \to 0} (6x^2 + 6xh + 2h^2)$
$= 6x^2 + 6x(0) + 2(0)^2$
$= 6x^2$

2. **a)** Question 1(b) shows that $f'(x) = 2x$. So, the gradient of the tangent to $f(x)$ at any point is $2x$.
If $x = -3$, then $f'(-3) = 2(-3) = -6$.
The equation of the tangent is $y = mx + c$.

$\therefore y = -6x + c$

If $x = -3$, then $f(-3) = (-3)^2 = 9$, so the tangent goes through the point $(-3; 9)$.

Substitute $(-3; 9)$ into the equation of the tangent:

$y = -6x + c$

$\therefore 9 = -6(-3) + c$

$\therefore c = -9$

Equation of the tangent to $f(x)$ that passes through $(-3; 9)$: $y = -6x - 9$

b) Question 1(c) shows that $f'(x) = -12x$. So, the gradient of the tangent to $f(x)$ at any point is $-12x$.

If $x = 1$, then $f'(1) = -12(1) = -12$.

The equation of the tangent is $y = mx + c$.

$\therefore y = -12x + c$

If $x = 1$, then $f(1) = -6(1)^2 = -6$, so the tangent goes through the point $(1; -6)$.

Substitute $(1; -6)$ into the equation of the tangent:

$y = -12x + c$

$\therefore -6 = -12(1) + c$

$\therefore c = 6$

Equation of the tangent to $f(x)$ that passes through $(1; -6)$: $y = -12x + 6$

c) Question 1(d) shows that $f'(x) = 4x + 5$. So, the gradient of the tangent to $f(x)$ at any point is $4x + 5$.

If $x = -4$, then $f'(-4) = 4(-4) + 5 = -11$.

The equation of the tangent is $y = mx + c$.

$\therefore y = -11x + c$

If $x = -4$ then $f(-4) = 2(-4)^2 + 5(-4) = 12$, so the tangent goes through the point $(-4; 12)$.

Substitute $(-4; 12)$ into the equation of the tangent:

$y = -11x + c$

$\therefore 12 = -11(-4) + c$

$\therefore c = -32$

Equation of the tangent to $f(x)$ that passes through $(-4; 12)$: $y = -11x - 32$

d) Question 1(e) shows that $f'(x) = -3x^2 + 1$. So, the gradient of the tangent to $f(x)$ at any point is $-3x^2 + 1$. If $x = 0$, then $f'(0) = -3(0)^2 + 1 = 1$.

The equation of the tangent is $y = mx + c$.

$\therefore y = x + c$

If $x = 0$, then $f(0) = -0^3 + 0 = 0$, so the tangent goes through the point (0; 0).
Substitute (0; 0) into the equation of the tangent:

$y = x + c$
$\therefore 0 = 0 + c$
$\therefore c = 0$

Equation of the tangent to $f(x)$ that passes through (0; 0): $y = x$

e) Question 1(f) shows that $f'(x) = 6x^2$. So, the gradient of the tangent to $f(x)$ at any point is $6x^2$. If $x = -2$, then $f'(-2) = 6(-2)^2 = 24$.
The equation of the tangent is $y = mx + c$.
$\therefore y = 24x + c$
If $x = -2$ then $f(-2) = 2(-2)^3 - 8 = -24$, so the tangent goes through the point (−2; −24).
Substitute (−2; −24) into the equation of the tangent:

$y = 24x + c$
$\therefore -24 = 24(-2) + c$
$\therefore c = 24$

Equation of the tangent to $f(x)$ that passes through (−2; −24): $y = 24x - 24$

Exercise 11.3 (page 175)

1. **a)** $f'(x) = 3$ **b)** $f'(x) = -20x^4$

c) $f'(x) = \frac{2}{4}x = \frac{1}{2}x$ **d)** $f'(x) = 0$

e) $f(x) = \frac{1}{x^2} = x^{-2}$
$\therefore f'(x) = (-2)x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$

f) $f(x) = -\frac{5}{x^7} = -5x^{-7}$
$\therefore f'(x) = (-7)(-5)x^{-7-1} = 35x^{-8} = \frac{35}{x^8}$

2. **a)** $y = \sqrt[3]{x} = x^{\frac{1}{3}}$
$\therefore \frac{dy}{dx} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

b) $y = \sqrt{x^5} = x^{\frac{5}{2}}$
$\therefore \frac{dy}{dx} = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{\frac{3}{2}}$

c) $y = -\sqrt{x} = -x^{\frac{1}{2}}$
$\therefore \frac{dy}{dx} = -\frac{1}{2}x^{\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{1}{2}} = -\frac{1}{2x^{\frac{1}{2}}}$

d) $y = \sqrt[4]{x^3} = x^{\frac{3}{4}}$
$\therefore \frac{dy}{dx} = \frac{3}{4}x^{\frac{3}{4}-1} = \frac{3}{4}x^{-\frac{1}{4}} = \frac{3}{4x^{\frac{1}{4}}}$

3. **a)** $\frac{d}{dx}\left(\frac{1}{2}\sin x\right) = \frac{1}{2}\cos x$

b) $\frac{d}{dx}(-5\cos x) = -5(-\sin x) = 5\sin x$

Exercise 11.4 (page 176)

1. **a)** $\frac{dy}{dx} = 3x^2 - 9$

b) $\frac{dy}{dx} = -12x^3 + 12x$

c) $\frac{dy}{dx} = 48x^5 + 5x^4 - 8x^3 + 6x + 12$

d) $y = \sqrt{x^5} - \sqrt[3]{x^2} + \sqrt[4]{x} = x^{\frac{5}{2}} - x^{\frac{2}{3}} + x^{\frac{1}{4}}$

$\therefore \frac{dy}{dx} = \frac{5}{2}x^{\frac{5}{2}-1} - \frac{2}{3}x^{\frac{2}{3}-1} + \frac{1}{4}x^{\frac{1}{4}-1}$

$= \frac{5}{2}x^{\frac{3}{2}} - \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{4}x^{-\frac{3}{4}}$

$= \frac{5}{2}x^{\frac{3}{2}} - \frac{2}{3x^{\frac{1}{3}}} + \frac{1}{4x^{\frac{3}{4}}}$

2. **a)** $\frac{d}{dx}(6\sin x - \cos x + x^2)$

$= 6\cos x - (-\sin x) + 2x$

$= 6\cos x + \sin x + 2x$

b) $\frac{d}{dx}\left(\frac{1}{3}x^6 - \frac{2}{5}x^5 + \frac{7}{8}x^4 - \frac{4}{9}x^3 - \frac{1}{12}x^2 + \sin x\right)$

$= \frac{6}{3}x^5 - \frac{10}{5}x^4 + \frac{28}{8}x^3 - \frac{12}{9}x^2 - \frac{2}{12}x + \cos x$

$= 2x^5 - 2x^4 + \frac{7}{2}x^3 - \frac{4}{3}x^2 - \frac{1}{6}x + \cos x$

c) $\frac{d}{dx}\left(\frac{2}{5}\sqrt{x^5} + \frac{2}{3}\sqrt{x^3} - \frac{1}{2}\sqrt[3]{x^4}\right)$

$= \frac{d}{dx}\left(\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{4}{3}}\right)$

$= x^{\frac{5}{2}-1} - x^{\frac{3}{2}-1} - \frac{2}{3}x^{\frac{4}{3}-1}$

$= x^{\frac{3}{2}} - x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{1}{3}}$

d) $\frac{d}{dx}(x^5 - 3\cos x + \sqrt{x^7} - 4\sin x - 2)$

$= \frac{d}{dx}(x^5 - 3\cos x + x^{\frac{7}{2}} - 4\sin x - 2)$

$= 5x^4 - 3(-\sin x) + \frac{7}{2}x^{\frac{7}{2}-1} - 4\cos x$

$= 5x^4 + 3\sin x + \frac{7}{2}x^{\frac{5}{2}} - 4\cos x$

Exercise 11.5 (page 177)

1. **a)** Let $f(x) = x^2 + x$ and $g(x) = 3x - 5$.

Then $f'(x) = 2x + 1$ and $g'(x) = 3$.

$(fg)' = fg' + gf'$

$\therefore \frac{dy}{dx} = (x^2 + x)(3) + (3x - 5)(2x + 1)$

$= 3x^2 + 3x + 6x^2 + 3x - 10x - 5$

$= 9x^2 - 4x - 5$

b) $y = (x^2 + x)(3x - 5)$
$= 3x^3 - 5x^2 + 3x^2 - 5x$
$= 3x^3 - 2x^2 - 5x$
$\therefore \frac{dy}{dx} = 9x^2 - 4x - 5$

c) The answers to a) and b) are the same.

2. **a)** $y = (\sqrt{x} + 2)(\sqrt{x} - 2)$
$= (x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$

Let $f(x) = x^{\frac{1}{2}} + 2$ and $g(x) = x^{\frac{1}{2}} - 2$.

Then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

$(fg)' = fg' + gf'$
$\therefore \frac{dy}{dx} = (x^{\frac{1}{2}} + 2)(\frac{1}{2}x^{-\frac{1}{2}}) + (x^{\frac{1}{2}} - 2)(\frac{1}{2}x^{-\frac{1}{2}})$
$= \frac{1}{2} + x^{-\frac{1}{2}} + \frac{1}{2} - x^{-\frac{1}{2}}$
$= 1$

b) $y = (\sqrt{x} + 2)(\sqrt{x} - 2)$
$= x - 4$ (Difference of two squares)
$\therefore \frac{dy}{dx} = 1$

c) The answers to a) and b) are the same.

3. **a)** $\frac{d}{dx}(x - 5)(x + 6)$; let $u(x) = x - 5$ and $v(x) = x + 6$.
Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 1$.
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$= (x - 5)(1) + (x + 6)(1)$
$= x - 5 + x + 6$
$= 2x + 1$

b) $\frac{d}{dx}(x + 2)(x^2 - 3)$; let $u(x) = x + 2$ and $v(x) = x^2 - 3$.
Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2x$.
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$= (x + 2)(2x) + (x^2 - 3)(1)$
$= 2x^2 + 4x + x^2 - 3$
$= 3x^2 + 4x - 3$

c) $\frac{d}{dx}(x^2 + x - 4)(3x + 2)$; let $u(x) = x^2 + x - 4$ and $v(x) = 3x + 2$.
Then $\frac{du}{dx} = 2x + 1$ and $\frac{dv}{dx} = 3$.

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$= (x^2 + x - 4)(3) + (3x + 2)(2x + 1)$
$= 3x^2 + 3x - 12 + 6x^2 + 3x + 4x + 2$
$= 9x^2 + 10x - 10$

d) $\frac{d}{dx}\left(\frac{1}{x} + x\right)\left(\frac{1}{x} - 2x\right)$; let $u(x) = \frac{1}{x} + x = x^{-1} + x$ and $v(x) = \frac{1}{x} - 2x = x^{-1} - 2x$.
Then $\frac{du}{dx} = -x^{-2} + 1$ and $\frac{dv}{dx} = -x^{-2} - 2$.
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$= (x^{-1} + x)(-x^{-2} - 2) + (x^{-1} - 2x)(-x^{-2} + 1)$
$= -x^{-3} - 2x^{-1} - x^{-1} - 2x - x^{-3} + x^{-1} + 2x^{-1} - 2x$
$= -2x^{-3} - 4x$
$= -\frac{2}{x^3} - 4x$

e) $\frac{d}{dx}\left(\sqrt{x}\right)(x - 1)$; let $u(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $v(x) = x - 1$.
Then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ and $\frac{dv}{dx} = 1$.
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$= \left(x^{\frac{1}{2}}\right)(1) + (x - 1)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$
$= x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$
$= \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2x^{\frac{1}{2}}}$

f) $\frac{d}{dx}\left(\sqrt[3]{x}\right)(x^2 + 5x)$; let $u(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ and $v(x) = x^2 + 5x$.
Then $\frac{du}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$ and $\frac{dv}{dx} = 2x + 5$.
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
$= \left(x^{\frac{1}{3}}\right)(2x + 5) + (x^2 + 5x)\left(\frac{1}{3}x^{-\frac{2}{3}}\right)$
$= 2x^{\frac{4}{3}} + 5x^{\frac{1}{3}} + \frac{1}{3}x^{\frac{4}{3}} + \frac{5}{3}x^{\frac{1}{3}}$
$= \frac{7}{3}x^{\frac{4}{3}} + \frac{20}{3}x^{\frac{1}{3}}$

g) $\frac{d}{dx}(\cos x - 2)(\sin x + \cos x + 3)$; let $u(x) = \cos x - 2$ and $v(x) = \sin x + \cos x + 3$.
Then $\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = \cos x - \sin x$.
$\frac{d}{dx}(uv)$
$= u\frac{dv}{dx} + v\frac{du}{dx}$
$= (\cos x - 2)(\cos x - \sin x) + (\sin x + \cos x + 3)(-\sin x)$
$= \cos^2 x - \sin x \cos x - 2\cos x + 2\sin x - \sin^2 x - \sin x \cos x - 3\sin x$
$= \cos^2 x - 2\sin x \cos x - 2\cos x - \sin x - \sin^2 x$

h) $\frac{d}{dx}(\cos x - \sqrt{x})^2$; let $u(x) = \cos x - \sqrt{x} = \cos x - x^{\frac{1}{2}}$ and $v(x) = \cos x - \sqrt{x} = \cos x - x^{\frac{1}{2}}$.

Then $\frac{du}{dx} = -\sin x - \frac{1}{2}x^{-\frac{1}{2}}$ and $\frac{dv}{dx} = -\sin x - \frac{1}{2}x^{-\frac{1}{2}}$.

$$\begin{aligned}\frac{d}{dx}(uv) &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= \left(\cos x - x^{\frac{1}{2}}\right)\left(-\sin x - \tfrac{1}{2}x^{-\frac{1}{2}}\right) \\ &\quad + \left(\cos x - x^{\frac{1}{2}}\right)\left(-\sin x - \tfrac{1}{2}x^{-\frac{1}{2}}\right) \\ &= -\sin x\cos x - \tfrac{1}{2}x^{-\frac{1}{2}}\cos x + x^{\frac{1}{2}}\sin x + \tfrac{1}{2} \\ &\quad - \sin x\cos x - \tfrac{1}{2}x^{-\frac{1}{2}}\cos x + x^{\frac{1}{2}}\sin x + \tfrac{1}{2} \\ &= -2\sin x\cos x - x^{-\frac{1}{2}}\cos x + 2x^{\frac{1}{2}}\sin x + 1 \\ &= -2\sin x\cos x - \frac{\cos x}{x^{\frac{1}{2}}} + 2x^{\frac{1}{2}}\sin x + 1\end{aligned}$$

Exercise 11.6 (page 179)

1. a) Let $f(x) = x^2 - x - 6$ and $g(x) = x - 3$.

Then $f'(x) = 2x - 1$ and $g'(x) = 1$.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(x-3)(2x-1) - (x^2 - x - 6)(1)}{(x-3)^2} \\ &= \frac{2x^2 - x - 6x + 3 - x^2 + x + 6}{(x-3)^2} \\ &= \frac{x^2 - 6x + 9}{(x-3)^2} \\ &= \frac{(x-3)^2}{(x-3)^2} \\ &= 1\end{aligned}$$

b) $y = \frac{x^2 - x - 6}{x - 3} = \frac{(x-3)(x+2)}{x-3} = x + 2$

$\therefore \frac{dy}{dx} = 1$

c) The answers to a) and b) are the same.

2. a) Let $f(x) = 6x^2 - x - 2$ and $g(x) = 2x + 1$.

Then $f'(x) = 12x - 1$ and $g'(x) = 2$.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(2x+1)(12x-1) - (6x^2 - x - 2)(2)}{(2x+1)^2} \\ &= \frac{24x^2 - 2x + 12x - 1 - 12x^2 + 2x + 4}{(2x+1)^2} \\ &= \frac{12x^2 + 12x + 3}{(2x+1)^2} \\ &= \frac{3(4x^2 + 4x + 1)}{(2x+1)^2}\end{aligned}$$

$= \frac{3(2x+1)^2}{(2x+1)^2}$

$= 3$

b) $y = \frac{6x^2 - x - 2}{2x + 1}$

$= \frac{(3x-2)(2x+1)}{2x+1}$

$= 3x - 2$

$\therefore \frac{dy}{dx} = 3$

c) The answers to a) and b) are the same.

3. **a)** $\frac{d}{dx}\left(\frac{1}{4-5x}\right)$; let $u(x) = 1$ and $v(x) = 4 - 5x$.

Then $\frac{du}{dx} = 0$ and $\frac{dv}{dx} = -5$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(4-5x)(0) - (1)(-5)}{(4-5x)^2}$$

$$= \frac{5}{(4-5x)^2}$$

b) $\frac{d}{dx}\left(\frac{2x+1}{2-x}\right)$; let $u(x) = 2x + 1$ and $v(x) = 2 - x$.

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = -1$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2-x)(2) - (2x+1)(-1)}{(2-x)^2}$$

$$= \frac{4 - 2x + 2x + 1}{(2-x)^2}$$

$$= \frac{5}{(2-x)^2}$$

c) $\frac{d}{dx}\left(\frac{x^5-4}{x^2-4}\right)$; let $u(x) = x^5 - 4$ and $v(x) = x^2 - 4$.

Then $\frac{du}{dx} = 5x^4$ and $\frac{dv}{dx} = 2x$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2-4)(5x^4) - (x^5-4)(2x)}{(x^2-4)^2}$$

$$= \frac{5x^6 - 20x^4 - 2x^6 + 8x}{(x^2-4)^2}$$

$$= \frac{3x^6 - 20x^4 + 8x}{(x^2-4)^2}$$

d) $\frac{d}{dx}\left(\frac{8}{x^2+2x-3}\right)$; let $u(x) = 8$ and $v(x) = x^2 + 2x - 3$.

Then $\frac{du}{dx} = 0$ and $\frac{dv}{dx} = 2x + 2$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2+2x-3)(0) - (8)(2x+2)}{(x^2+2x-3)^2}$$

$$= \frac{-16x - 16}{(x^2+2x-3)^2}$$

e) $\frac{d}{dx}\left(\frac{7x+2}{2-\sqrt{x}}\right)$; let $u(x) = 7x + 2$
and $v(x) = 2 - \sqrt{x} = 2 - x^{\frac{1}{2}}$.
Then $\frac{du}{dx} = 7$ and $\frac{dv}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{\left(2 - x^{\frac{1}{2}}\right)(7) - (7x+2)\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)}{(2 - x^{\frac{1}{2}})^2}$$
$$= \frac{14 - 7x^{\frac{1}{2}} + \frac{7}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{(2 - x^{\frac{1}{2}})^2}$$
$$= \frac{14 - \frac{7}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{(2 - x^{\frac{1}{2}})^2}$$
$$= \frac{14x^{\frac{1}{2}} - \frac{7}{2}x + 1}{x^{\frac{1}{2}}(2 - x^{\frac{1}{2}})^2}$$

f) $\frac{d}{dx}\left(\frac{5+\sin x}{\cos x - 3}\right)$; let $u(x) = 5 + \sin x$ and $v(x) = \cos x - 3$.
Then $\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = -\sin x$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(\cos x - 3)(\cos x) - (5 + \sin x)(-\sin x)}{(\cos x - 3)^2}$$
$$= \frac{\cos^2 x - 3\cos x + 5\sin x + \sin^2 x}{(\cos x - 3)^2}$$

4. $\tan x = \frac{\sin x}{\cos x}$
Let $f(x) = \sin x$ and $g(x) = \cos x$.
Then $f'(x) = \cos x$ and $g'(x) = -\sin x$.

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$
$$\therefore \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$
$$= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos x^2}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} \quad \text{(Use the identity } \cos^2 x + \sin^2 x = 1.\text{)}$$

Exercise 11.7 (page 181)

1. a) Perimeter of rectangle $= 2l + 2b$
$2x + 2b = 200$
$\therefore x + b = 100$
$\therefore b = (100 - x)$ m

b) Area of the rectangle $= l \times b$
$\therefore$ Area $= x(100 - x)$
$= (100x - x^2)$ m^2

c) The area will be at a maximum where $\frac{dA}{dx} = 0$.

$$A = 100x - x^2$$
$$\therefore \frac{dA}{dx} = 100 - 2x$$
$$\therefore 100 - 2x = 0$$
$$\therefore 100 = 2x$$
$$\therefore x = 50 \text{ m}$$

and, $y = 100 - 50 = 50$ m.

So, the rectangle will have a maximum area if it is a square with sides of 50 m.

2. a) The graph is a parabola.

b) The height will be a maximum where $h'(s) = 0$.

$$h(s) = -\frac{1}{40}s^2 + s$$
$$\therefore h'(s) = -\frac{2}{40}s + 1$$
$$= -\frac{1}{20}s + 1$$
$$\therefore -\frac{1}{20}s + 1 = 0$$
$$\therefore \frac{1}{20}s = 1$$
$$\therefore s = 20 \text{ m}$$
$$h(20) = -\frac{1}{40}(20)^2 + 20$$
$$= -\frac{1}{40}(400) + 20$$
$$= -10 + 20$$
$$= 10 \text{ m}$$

The maximum height of the ball above the ground was 10 m.

c) On the ground, $h(s) = 0$.

$$\therefore -\frac{1}{40}s^2 + s = 0$$
$$\therefore s^2 - 40s = 0$$
$$\therefore s(s - 40) = 0$$
$$\therefore s = 0 \text{ or } s = 40$$

The ball had travelled a horizontal distance of 40 m by the time it reached the ground again.

3. a) The height will be a minimum where $h'(s) = 0$.

$$h(s) = \frac{4}{1\,125}s^2 - \frac{8}{15}s + 50$$
$$\therefore h'(s) = \frac{8}{1\,125}s - \frac{8}{15}$$
$$\therefore \frac{8}{1\,125}s - \frac{8}{15} = 0$$
$$\therefore \frac{8}{1\,125}s = \frac{8}{15}$$
$$\therefore s = \frac{8}{15} \times \frac{1\,125}{8}$$
$$= 75 \text{ m}$$

$h(50) = \frac{4}{1\,125}(75)^2 - \frac{8}{15}(75) + 50 = 30$
So, the minimum height of the bridge above the water is 30 m.

b) The turning point of the parabola is (75; 30).
75 m corresponds to the midpoint of the river.
$\therefore w = 75 \times 2 = 150$ m
The width of the river is 150 m.

4. a) $1 \text{ m}^3 = 1\,000\ \ell$
$\therefore 8 \text{ m}^3 = 8\,000\ \ell$
$\therefore \text{Volume} = 8 \text{ m}^3$
$\therefore \pi r^2 \times h = 8$
$\therefore h = \frac{8}{\pi r^2}$

b) $\text{Area} = 2 \times \pi r^2 + 2\pi r \times h = 2\pi r^2 + 2\pi r \times \frac{8}{\pi r^2} = 2\pi r^2 + \frac{16}{r}$

c) The least amount of steel will be needed when the total surface area of the tank is a minimum (where $\frac{dA}{dr} = 0$).

$$\begin{aligned} \text{Area} &= 2\pi r^2 + \frac{16}{r} \\ &= 2\pi r^2 + 16r^{-1} \\ \therefore \frac{dA}{dr} &= 4\pi r - 16r^{-2} \end{aligned}$$

$$\begin{aligned} \therefore 4\pi r - 16r^{-2} &= 0 \\ \therefore 4\pi r - \frac{16}{r^2} &= 0 \\ \therefore 4\pi r &= \frac{16}{r^2} \\ \therefore 4\pi r^3 &= 16 \\ \therefore \pi r^3 &= 4 \\ \therefore r^3 &= \frac{4}{\pi} \\ \therefore r &= \sqrt[3]{\frac{4}{\pi}} \\ &= 1.0839 \text{ m} \end{aligned}$$

$$\begin{aligned} \therefore h &= \frac{8}{\pi r^2} \\ &= \frac{8}{\pi(1.0839)^2} \\ &= 2.1675 \text{ m} \end{aligned}$$

The radius of the tank must be 1.0839 m and the height of the tank must be 2.1675 m.

d) The height of the tank is almost exactly twice the radius of the tank.

e) $\text{Area} = 2\pi r^2 + \frac{16}{r} = 2\pi(1.0839)^2 + \frac{16}{1.0839} = 22.14 \text{ m}^2$
23 m^2 of steel will be needed to build the water tank.

5. **a)** Let $y = \frac{1}{3}x^3 - \frac{35}{2}x^2 + 250x + 900$

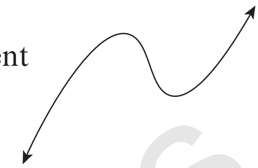
Find the maximum and minimum values where $\frac{dy}{dx} = 0$.

$\therefore x^2 - 35x + 250 = 0$

$\therefore (x - 10)(x - 25) = 0$

$\therefore x = 10$ or $x = 25$

This cubic function has the shape shown on the right (as the coefficient of x^3 is positive).



The maximum value of the share was reached on 10 January.

b) The minimum value of the share was reached on 25 January.

c) **i)** For the minimum value, $x = 25$

Minimum value

$= \frac{1}{3}(25)^3 - \frac{35}{2}(25)^2 + 250(25) + 900 =$ ₦1 420.83

ii) For the maximum value, $x = 10$

Maximum value

$= \frac{1}{3}(10)^3 - \frac{35}{2}(10)^2 + 250(10) + 900 =$ ₦1 983.33

Exercise 11.8 (page 184)

1. **a)** 5 minutes = 300 seconds

5(300) = 2(300) = 600

The jogger will run 600 m in 5 minutes.

b) 5 km = 5 000 m

$\therefore 2t = 5\,000$

$\therefore t = 2\,500$

2 500 s = 41 min. and 40 s

The jogger will take 41 minutes and 40 seconds to run 5 km.

c) $v(t) = s'(t) = 2$ m/s

d) $a(t) = v'(t) = 0$ m/s^2

e) The velocity of the jogger is a constant (2 m/s). This means that his jogging rate is constant. He maintains the same speed throughout his run.

The jogger's acceleration is 0 m/s^2. This means that he does not accelerate (go faster) or decelerate (go slower) at any point during his run.

2. a) $v(t) = s'(t) = (3)\left(\frac{1}{3}\right)t^2 - 2(8)t + 64 = t^2 - 16t + 64$

b) $a(t) = v'(t) = 2t - 16$

c) The initial velocity of the car is velocity of the car when $t = 0$.
$v(0) = 0^2 - 16(0) + 64 = 64$ m/s

d) 64 m/s = (0.064 × 60 × 60) km/h = 230.4 km/h

e) Average velocity $= \frac{\text{change in distance}}{\text{change in time}}$

$= \frac{s_2 - s_1}{t_2 - t_1}$

$= \frac{s(3) - s(2)}{3 - 2}$

$= \frac{\left[\frac{1}{3}(3)^3 - 8(3)^2 + 64(3)\right] - \left[\frac{1}{3}(2)^3 - 8(2)^2 + 64(2)\right]}{3 - 2}$

$= \frac{129 - 98.67}{1}$

$= 30.33$ m/s

f) Instantaneous velocity is given by $v(t) = t^2 - 16t + 64$.
i) $v(2) = 2^2 - 16(2) + 64 = 36$ m/s
ii) $v(3) = 3^2 - 16(3) + 64 = 25$ m/s

g) $a(t) = 2t - 16$
i) $a(2) = 2(2) - 16 = -12$ m/s^2
ii) $a(3) = 2(3) - 16 = -10$ m/s^2

h) The car will come to a complete stop when the velocity reaches 0.
$\therefore t^2 - 16t + 64 = 0$
$\therefore (t - 8)(t - 8) = 0$
$\therefore t = 8$ seconds
It will take 8 seconds for the car to come to a complete stop.

i) The stopping distance (the distance the car travelled in 8 seconds)
$= \frac{1}{3}(8)^3 - 8(8)^2 + 64(8)$
$= 170.67$ m

3. a) Average velocity

$= \frac{\text{change in distance}}{\text{change in time}}$

$= \frac{s_2 - s_1}{t_2 - t_1}$

$= \frac{s(7) - s(6)}{7 - 6}$

$= \frac{(4(7)^2 - 16(7) + 63) - (4(6)^2 - 16(6) + 63)}{1}$

$= \frac{147 - 111}{1}$

$= 36$ m/s

b) Instantaneous velocity is given by $v(t) = s'(t) = 8t - 16$.
$s'(6) = 8(6) - 16 = 32$ m/s

c) Acceleration is given by $a(t) = v'(t) = 8$.
$a(2) = 8$ m/s^2

4. a) The rate of change of the temperature is given by $\frac{dT}{dt}$.

$T = -0.005t^3 - 0.2t + 30$

$\therefore \frac{dT}{dt} = -0.015t^2 - 0.2$

i) When $t = 1$ minute:
$\frac{dT}{dt} = -0.015(1)^2 - 0.2 = -0.215$ °C/min.

ii) When $t = 5$ minutes:
$\frac{dT}{dt} = -0.015(5)^2 - 0.2 = -0.575$ °C/min.

iii) When $t = 8$ minutes:
$\frac{dT}{dt} = -0.015(8)^2 - 0.2 = -1.16$ °C/min.

iv) When $t = 10$ minutes:
$\frac{dT}{dt} = -0.015(10)^2 - 0.2 = -1.7$ °C/min.

b) $T = -0.005(10)^3 - 0.2(10) + 30 = 23$ °C
The resulting temperature of the room is 23 °C.

Assess your progress (page 185)

1. a) $\lim_{x \to 6} \frac{x^2 - 36}{x - 6} = \lim_{x \to 6} \frac{(x + 6)(x - 6)}{x - 6}$

$= \lim_{x \to 6} (x + 6) = 12$

b) $\lim_{x \to 0} \frac{x^2 - 3x}{x^2 + 9x} = \lim_{x \to 0} \frac{x(x - 3)}{x(x + 9)} = \lim_{x \to 0} \frac{x - 3}{x + 9}$

$= \frac{0 - 3}{0 + 9} = -\frac{1}{3}$

c) $\lim_{x \to -4} \frac{x^2 + 2x - 8}{x + 4} = \lim_{x \to -4} \frac{(x + 4)(x - 2)}{x + 4}$

$= \lim_{x \to -4} (x - 2) = -4 - 2 = -6$

2. a) $f(x) = 2x + 1$

$\therefore f(x + h) = 2(x + h) + 1 = 2x + 2h + 1$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{2x + 2h + 1 - (2x + 1)}{h}$

$= \lim_{h \to 0} \frac{2x + 2h + 1 - 2x - 1}{h}$

$= \lim_{h \to 0} \frac{2h}{h}$

$= \lim_{h \to 0} 2$

$= 2$

b) $f(x) = -x^3 + 5x$

$\therefore f(x + h)$

$= -(x + h)^3 + 5(x + h)$

$= -(x + h)(x + h)^2 + 5x + 5h$

$= -(x + h)(x^2 + 2xh + h^2) + 5x + 5h$

$= -(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) + 5x + 5h$

$= -x^3 - 2x^2h - xh^2 - x^2h - 2xh^2 - h^3 + 5x + 5h$

$= -x^3 - 3x^2h - 3xh^2 - h^3 + 5x + 5h$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{-x^3 - 3x^2h - 3xh^2 - h^3 + 5x + 5h - (-x^3 + 5x)}{h}$

$= \lim_{h \to 0} \frac{-x^3 - 3x^2h - 3xh^2 - h^3 + 5x + 5h + x^3 - 5x}{h}$

$= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3 + 5h}{h}$

$= \lim_{h \to 0} (-3x^2 - 3xh - h^2 + 5)$

$= -3x^2 - 3x(0) - 0^2 + 5$

$= -3x^2 + 5$

3. **a)** $f(x) = x^3 + x - 9$

$\therefore f'(x) = 3x^2 + 1$

If $x = 0$, then $f'(0) = 1$.

The equation of the tangent is $y = mx + c$.

$\therefore y = x + c$

If $x = 0$, then $f(0) = -9$; so the tangent goes through the point $(0; -9)$.

Substitute $(0; -9)$ into the equation of the tangent.

$y = x + c$

$\therefore -9 = 0 + c$

$\therefore c = -9$

Equation of the tangent to $f(x)$ passes through $(0; -9)$: $y = x - 9$

b) $f'(x) = 3x^2 + 1$

If $x = -1$, then $f'(-1) = 3(-1)^2 + 1 = 4$.

The equation of the tangent is $y = mx + c$.

$\therefore y = 4x + c$

If $x = -1$, then $f(-1) = (-1)^3 + (-1) - 9 = -11$, so the tangent goes through $(-1; -11)$.

Substitute $(-1; -11)$ into the equation of the tangent:

$y = 4x + c$

$\therefore -11 = 4(-1) + c$

$\therefore c = -7$

Equation of the tangent to $f(x)$ that passes through $(-1; -11)$: $y = 4x - 7$

4. a) $\frac{dy}{dx} = 36x^3 + 6x^2 - 2x + 4$

b) $y = \frac{4}{x} - \frac{5}{x^2} + \frac{1}{3x^3} = 4x^{-1} - 5x^{-2} + \frac{1}{3}x^{-3}$

$\therefore \frac{dy}{dx} = -4x^{-2} + 10x^{-3} - x^{-4}$

c) $y = \sqrt[4]{x} - 4 = x^{\frac{1}{4}} - 4$

$\therefore \frac{dy}{dx} = \frac{1}{4}x^{-\frac{3}{4}}$

d) $y = (\sqrt{x} + 3)(\sqrt{x} - 3) = x - 9$ (Difference of squares.)

$\therefore \frac{dy}{dx} = 1$

e) Let $f(x) = x + 6$ and $g(x) = 2x^2 - 1$

Then $f'(x) = 1$ and $g'(x) = 4x$

$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$\therefore \frac{dy}{dx} = \frac{(2x^2 - 1)(1) - (x + 6)(4x)}{2x^2 - 1}$

$= \frac{2x^2 - 1 - x - 6}{(2x - 1)^2}$

$= \frac{2x^2 - x - 7}{(2x - 1)^2}$

f) Let $f(x) = x^3 - 1$ and $g(x) = \sin x$.

Then $f'(x) = 3x^2$ and $g'(x) = \cos x$.

$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

$\therefore \frac{dy}{dx} = \frac{(\sin x)(3x^2) - (x^3 - 1)(\cos x)}{(\sin x)^2}$

$= \frac{3x^2 \sin x - x^3 \cos x + \cos x}{\sin^2 x}$

5. a) $\frac{d}{dx}(\sin x - 4\cos x + 5x^3) = \cos x + 4\sin x + 15x^2$

b) Let $u(x) = 1 + \cos x$ and $v(x) = \sin x - 1$.

Then, $\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = \cos x$.

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (1 + \cos x)(\cos x) + (\sin x - 1)(-\sin x)$

$= \cos x + \cos^2 x - \sin^2 x + \sin x$

c) $\frac{d}{dx}(x^2 + 2x - 3)(4x + 1)$

$= \frac{d}{dx}(4x^3 + x^2 + 8x^2 + 2x - 12x - 3)$

$= \frac{d}{dx}(4x^3 + 9x^2 - 10x - 3)$

$= 12x^2 + 18x - 10$

d) $\frac{d}{dx}\left(\frac{3}{10}\sqrt{x^5} + \frac{5}{6}\sqrt{x^3} - \frac{3}{2}\sqrt[3]{x^2}\right)$

$= \frac{d}{dx}\left(\frac{3}{10}x^{\frac{5}{2}} + \frac{5}{6}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{2}{3}}\right)$

$= \left(\frac{5}{2}\right)\left(\frac{3}{10}\right)x^{\frac{5}{2}-1} + \left(\frac{3}{2}\right)\left(\frac{5}{6}\right)x^{\frac{3}{2}-1} - \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{\frac{2}{3}-1}$

$= \frac{3}{4}x^{\frac{3}{2}} + \frac{5}{4}x^{\frac{1}{2}} - x^{-\frac{1}{3}}$

$= \frac{3}{4}x^{\frac{3}{2}} + \frac{5}{4}x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{3}}}$

e) Let $u(x) = 3$ and $v(x) = 2 - 7x$.
Then, $\frac{du}{dx} = 0$ and $\frac{dv}{dx} = -7$.

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2-7x)(0) - (3)(-7)}{(2-7x)^2}$

$= \frac{21}{(2-7x)^2}$

f) Let $u(x) = 6x + 1$ and $v(x) = 5 - 2\sqrt{x} = 5 - 2x^{\frac{1}{2}}$.
Then, $\frac{du}{dx} = 6$ and $\frac{dv}{dx} = -\left(\frac{1}{2}\right)(2)x^{-\frac{1}{2}} = -x^{-\frac{1}{2}}$.

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(5-2x^{\frac{1}{2}})(6) - (6x+1)(-x^{-\frac{1}{2}})}{(5-2x^{\frac{1}{2}})^2}$

$= \frac{30 - 12x^{\frac{1}{2}} + 6x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{(5-2x^{\frac{1}{2}})^2}$

$= \frac{30 - 6x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}}}{(5-2x^{\frac{1}{2}})^2}$

6. **a)** Let $y = -2x^3 + 30x^2 - 96x + 9\,000$
Find the maximum and minimum values where $\frac{dy}{dx} = 0$.

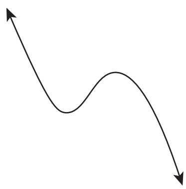
$\therefore -6x^2 + 60x - 96 = 0$

$\therefore x^2 - 10x + 16 = 0$

$\therefore (x-2)(x-8) = 0$

$\therefore x = 2$ or $x = 8$

This cubic function has the shape shown on the right (as the coefficient of x^3 is negative).



i) the second Friday
ii) the eighth Friday

b) **i)** For the minimum value, $x = 2$:
$\therefore y = -2(2)^3 + 30(2)^2 - 96(2) + 9\,000 = 8\,912$
So, minimum value $= 8\,912 \times$ ₦1 000
$=$ ₦8 912 000

ii) For the maximum value, $x = 9$:
$\therefore y = -2(8)^3 + 30(8)^2 - 96(8) + 9\,000 = 9\,128$

So, maximum value = 9 128 × ₦1 000
= ₦9 128 000

7. The profit will be a maximum where $\frac{dP}{dx} = 0$.

$$P = -\frac{8}{3}x^3 + 1\,800x + 29\,500$$
$$\frac{dP}{dx} = -(3)\left(\frac{8}{3}\right)x^2 + 1\,800$$
$$\frac{dP}{dx} = -8x^2 + 1\,800$$
$$\therefore -8x^2 + 1\,800 = 0$$
$$\therefore 8x^2 = 1\,800$$
$$\therefore x^2 = 225$$
$$\therefore x = \pm\sqrt{225}$$
$$= \pm 15$$

But, the factory cannot employ a negative number of people, so the factory should employ 15 employees in order to make the maximum possible profit.

TOPIC 12 Integration of simple algebraic functions

Introduction

In Topic 11, your students were introduced to calculus for the first time as they learnt about limits and differentiation. In this topic, they will learn about the integration of simple algebraic functions.

The idea of integration is more abstract than differentiation and more difficult to relate to the real world. The most important aspect for your students to grasp is that integration is the reverse of differentiation, and differentiation is the reverse of integration. Your students will learn what it means to integrate an algebraic function, how to use the standard integrals of a few basic functions, how to apply the method of integration by substitution and the method of integration by partial fractions and they will discover a few practical uses of integration. They will use definite integrals to find the area under a curve, use Simpson's rule to approximate the area under the curve of a polynomial and apply integration to solving real-life situations.

Common difficulties

Common difficulties students experience include:

- Students forget to write + c at the end of their answers when working with the indefinite integral. Remind them to do so until it becomes a habit.
- One of the most common problems that students find when doing integration is that they are unsure exactly how to use each kind of notation. For example, when integrating a function, they are not sure when they must write $\int$ and dx and at what stage they must no longer write it. Explain that they must write $\int$ and dx while they are simplifying or manipulating the form of the function to be integrated. When they integrate the function, they must no longer include $\int$ and dx, but write + c at the end when working with an indefinite integral.

- When doing integration, your students will often work with common fractions. In particular, they need to be able to add, subtract and multiply common fractions. Even at this level, some students will not be confident about calculations with fractions. Remediate any problems as they arise. If many students struggle with fractions, work through a few examples with the whole class.
- As always, advise your class to be very careful when multiplying different signs. Mistakes with signs can occur easily and will cost your students unnecessary marks in tests and examinations.
- In general, advise your students to use differentiation to check their answers to integration problems. (The only exception is when they integrate by partial fractions, because they have not yet learnt how to differentiate the *ln* function.) By differentiating their answers, they will be able to pick up any mistakes that they might have made. Another advantage to this checking process is that they will have additional practice doing differentiation.

Preparation

Prepare the following posters to display in your classroom:

- the different rules used in integration
- the different notations we use when doing integration as well as examples of how to use each one
- examples of integration by substitution
- examples of integration by partial fractions
- Simpson's rule, with an example of how it is used
- the different formulae that are used in this topic
- the basic laws of exponents, with examples.

Introducing students to the topic

Remind your students that they have learnt about differentiation in Topic 11. Tell them that they will learn about integration for the first time in this topic. Tell them that differentiation and integration are the two main pillars on which all calculus is built, and that they are reverses of one another.

Tell students that those who will pursue future studies involving the sciences or engineering will find that calculus forms a key part of their studies and they will use differentiation and integration regularly. Refer to the posters you prepared and have displayed in your classroom and tell students that these posters summarise the key information that they will need as they work through this topic.

Answers

Exercise 12.1 (page 189)

1. a) $\int x^3\,dx = \frac{x^4}{4} + c$ b) $\int x^4\,dx = \frac{x^5}{5} + c$

   c) $\int x\,dx = \frac{x^2}{2} + c$ d) $\int x^7\,dx = \frac{x^8}{8} + c$

   e) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$

   $= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

   $= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

   $= \frac{2x^{\frac{3}{2}}}{3} + c$

   f) $\int \sqrt[3]{x^2}\,dx = \int x^{\frac{2}{3}}\,dx$

   $= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + c$

   $= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c$

   $= \frac{3x^{\frac{5}{3}}}{5} + c$

2. Students check their answers to question 1 by finding the derivative of each answer. They should find that integration and differentiation are inverses of one another.

Exercise 12.2 (page 190)

Students should use differentiation to check their answers.

1. $\int 4x^3\,dx = \frac{4x^4}{4} + c = x^4 + c$

2. $\int 2x^4\,dx = \frac{2x^5}{5} + c$

3. $\int (x + 5x^2)\,dx = \frac{x^2}{2} + \frac{5x^3}{3} + c$

4. $\int (2 - 3x)\,dx = 2x - \frac{3x^2}{2} + c$

5. $\int (7x^6 + x^5 - 3x)\,dx = 7\frac{x^7}{7} + \frac{x^6}{6} - \frac{3x^2}{2} + c = x^7 + \frac{x^6}{6} - \frac{3x^2}{2} + c$

6. $\int (x^6 - 6x^5 + 14)\,dx = \frac{x^7}{7} - \frac{6x^6}{6} + 14x + c = \frac{x^7}{7} - x^6 + 14x + c$

7. $\int(2\sqrt{x} - 4x^5)\,dx$

$= \int(2x^{\frac{1}{2}} - 4x^5)\,dx$

$= \frac{2x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{4x^6}{6} + c$

$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^6}{3} + c$

$= \frac{4x^{\frac{3}{2}}}{3} - \frac{2x^6}{3} + c$

8. $\int\left(\frac{7}{2}x^6 + \frac{1}{2}x\right)dx$

$= \frac{7}{2}\left(\frac{x^7}{7}\right) + \frac{1}{2}\left(\frac{x^2}{2}\right) + c$

$= \frac{x^7}{2} + \frac{x^2}{4} + c$

9. $\int\left(\frac{x^5}{2} + 0.2x^2 + \sqrt[3]{x}\right)dx$

$= \int\left(\frac{1}{2}x^5 + \frac{1}{5}x^2 + x^{\frac{1}{3}}\right)dx$

$= \frac{1}{2}\left(\frac{x^6}{6}\right) + \frac{1}{5}\left(\frac{x^3}{3}\right) + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c$

$= \frac{x^6}{12} + \frac{x^3}{15} + \frac{3x^{\frac{4}{3}}}{4} + c$

10. $\int4\left(\frac{3x^8}{4} + 25x^4 + \frac{4x^3}{9}\right)dx$

$= \int\left(3x^8 + 100x^4 + \frac{16x^3}{9}\right)dx$

$= \frac{3x^9}{9} + \frac{100x^5}{5} + \frac{16x^4}{36} + c$

$= \frac{x^9}{3} + 20x^5 + \frac{4x^4}{9} + c$

Exercise 12.3 (page 191)

Students should use differentiation to check their answers.

1. $\int\frac{1}{3}\,dx = \frac{1}{3}x + c$

2. $\int2x^7\,dx = \frac{2x^8}{8} + c = \frac{x^8}{4} + c$

3. $\int(x - 4)^2\,dx = \int(x^2 - 8x + 16)\,dx$

$= \frac{x^3}{3} - \frac{8x^2}{2} + 16x + c$

$= \frac{x^3}{3} - 4x^2 + 16x + c$

4. $\int(\sin x + 2\cos x + 3)\,dx = -\cos x + 2\sin x + c$

5. $\int\left(2\sin x + \frac{x^3}{4}\right)dx = -2\cos x + \frac{x^4}{16} + c$

6. $\int\left(\frac{1}{x^3} - \cos x\right)dx = \int(x^{-3} - \cos x)\,dx$

$= \frac{x^{-2}}{-2} - \sin x + c$

$= -\frac{1}{2x^2} - \sin x + c$

7. $\int(5\sqrt{x} - \sqrt[4]{x^5} - 15)\,dx$

$= \int\left(5x^{\frac{1}{2}} - x^{\frac{5}{4}} - 15\right)dx$

$= \frac{5x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{9}{4}}}{\frac{9}{4}} - 15x + c$

$= \frac{10x^{\frac{3}{2}}}{3} - \frac{4x^{\frac{9}{4}}}{9} - 15x + c$

8. $\int\left[x^2\left(\frac{1}{x^2} - \frac{3}{x^6} + 9\right)\right]dx$

$= \int\left(1 - \frac{3}{x^4} + \frac{9}{x^2}\right)dx$

$= \int(1 - 3x^{-4} + 9x^{-2})\,dx$

$= x - \frac{3x^{-3}}{-3} + \frac{9x^{-1}}{-1} + c$

$= x + x^{-3} - 9x^{-1} + c$

$= x + \frac{1}{x^3} - \frac{9}{x} + c$

9. $\int\left(x^6 + \frac{12}{5}x^5 + \frac{2}{3}x^4\right)dx$

$= \frac{x^7}{7} + \frac{12}{5}\left(\frac{x^6}{6}\right) + \frac{2}{3}\left(\frac{x^5}{5}\right) + c$

$= \frac{x^7}{7} + \frac{2x^6}{5} + \frac{2x^5}{15} + c$

10. $\int\frac{\frac{6}{x} + \frac{3}{x^4} - 6x}{3x}\,dx$

$= \int\left(\frac{2}{x^2} + \frac{1}{x^5} - 2\right)dx$

$= \int(2x^{-2} + x^{-5} - 2)\,dx$

$= -2x^{-1} + \frac{x^{-4}}{-4} - 2x + c$

$= -\frac{2}{x} - \frac{1}{4x^4} - 2x + c$

Exercise 12.4 (page 193)

1. Let $u = 2x$. Then $\frac{du}{dx} = 2$.

$\int(2\sin 2x)\,dx = \int(\sin 2x) \times 2\,dx$

$= \int\sin u\,\frac{du}{dx}\,dx$

$= \int\sin u\,du$

$= -\cos u + c$

$= -\cos 2x + c$

2. Let $u = x^3 + 1$. Then $\frac{du}{dx} = 3x^2$.

$\int x^2(x^3 + 1)\,dx = \frac{1}{3}\int(x^3 + 1)3x^2\,dx$

$= \frac{1}{3}\int u\,\frac{du}{dx}\,dx$

$= \frac{1}{3}\int u\,du$

$= \frac{1}{3}\,\frac{u^2}{2} + c$

$= \frac{1}{6}u^2 + c$

$= \frac{1}{6}(x^3 + 1)^2 + c$

3. Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

$\int \sin x \times \cos x \, dx = \int u \frac{du}{dx} \, dx$

$= \int u \, du$

$= \frac{u^2}{2} + c$

$= \frac{1}{2}\sin^2 x + c$

4. Let $u = 5x$. Then $\frac{du}{dx} = 5$.

$\int \cos 5x \, dx = \frac{1}{5}\int (\cos 5x)5 \, dx$

$= \frac{1}{5}\int \cos u \frac{du}{dx} \, dx$

$= \frac{1}{5}\int \cos u \, du$

$= \frac{1}{5}\sin u + c$

$= \frac{1}{5}\sin 5x + c$

5. Let $u = x^3 + x$. Then $\frac{du}{dx} = 3x^2 + 1$.

$\int (x^3 + x)^9(3x^2 + 1) \, dx = \int u^9 \frac{du}{dx} \, dx$

$= \int u^9 \, du$

$= \frac{u^{10}}{10} + c$

$= \frac{1}{10}(x^3 + x)^{10} + c$

6. Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

$\int \sin^5 x \times \cos x \, dx = \int u^5 \frac{du}{dx} \, dx$

$= \int u^5 \, du$

$= \frac{u^6}{6} + c$

$= \frac{1}{6}\sin^6 x + c$

Exercise 12.5 (page 195)

1. $\int \frac{4}{(x^2 + 4x + 3)} \, dx = \int \frac{4}{(x + 3)(x + 1)} \, dx$

Let $\frac{4}{(x + 3)(x + 1)} = \frac{A}{(x + 3)} + \frac{B}{(x + 1)})$

$= \frac{A(x + 1) + B(x + 3)}{(x + 3)(x + 1)}$

$\therefore \frac{4}{(x + 3)(x + 1)} = \frac{Ax + A + Bx + 3B)}{(x + 3)(x + 1)}$

$\therefore Ax + A + Bx + 3B = 4$

$\therefore (A + B)x + (A + 3B) = 4$

$\therefore A + B = 0$ and $A + 3B = 4$

$A + B = 0$

$\therefore A = -B$ ①

$A + 3B = 4$ ②

Substitute equation ① into equation ②:

$-B + 3B = 4$

$\therefore 2B = 4$

$\therefore B = 2$ ③

Substitute equation ③ into equation ①:

$A = -B = -2$

$\therefore A = -2$ and $B = 2$

So now we have:

$$\frac{4}{(x + 3)(x + 1)} = \frac{-2}{x + 3} + \frac{2}{x + 1}$$

$$\therefore \int \frac{4}{(x + 3)(x + 1)}\, dx = \int \left(\frac{-2}{x + 3} + \frac{2}{x + 1}\right) dx$$

$$= -2\int \frac{1}{x + 3}\, dx + 2\int \frac{1}{x + 1}\, dx$$

$$= -2\, ln\,(x + 3) + 2\, ln\,(x + 1) + c$$

2. $$\int \frac{x}{x^2 - 6x + 8}\, dx = \int \frac{x}{(x - 2)(x - 4)}\, dx$$

Let $\frac{x}{(x - 2)(x - 4)} = \frac{A}{x - 2} + \frac{B}{x - 4}$

$$= \frac{A(x - 4) + B(x - 2)}{(x - 2)(x - 4)}$$

$$\therefore \frac{x}{(x - 2)(x - 4)} = \frac{Ax - 4A + Bx - 2B)}{(x - 2)(x - 4)}$$

$\therefore Ax - 4A + Bx - 2B = x$

$\therefore (A + B)x + (-4A - 2B) = x$

$\therefore A + B = 1$ and $-4A - 2B = 0$

$A + B = 1$

$\therefore A = 1 - B$ ①

$-4A - 2B = 0$

$\therefore 2A + B = 0$ ②

Substitute equation ① into equation ②:

$2(1 - B) + B = 0$

$\therefore 2 - 2B + B = 0$

$\therefore B = 2$ ③

Substitute equation ③ into equation ①:

$A = 1 - B = 1 - 2 = -1$

$\therefore A = -1$ and $B = 2$

So now we have:

$$\frac{x}{(x - 2)(x - 4)} = \frac{-1}{x - 2} + \frac{2}{x - 4}$$

$$\therefore \int \frac{x}{(x - 2)(x - 4)}\, dx = \int \left(\frac{-1}{x - 2} + \frac{2}{x - 4}\right) dx$$

$$= -\int \frac{1}{x - 2}\, dx + 2\int \frac{1}{x - 4}\, dx$$

$$= -ln\,(x - 2) + 2\, ln\,(x - 4) + c$$

3. $\int \frac{x + 14}{x^2 + 7x + 10}\, dx = \int \frac{x + 14}{(x + 2)(x + 5)}\, dx$

Let $\frac{x + 14}{(x + 2)(x + 5)} = \frac{A}{x + 2} + \frac{B}{x + 5}$

$= \frac{A(x + 5) + B(x + 2)}{(x + 2)(x + 5)}$

$\therefore \frac{x + 14}{(x + 2)(x + 5)} = \frac{Ax + 5A + Bx + 2B)}{(x + 2)(x + 5)}$

$\therefore Ax + 5A + Bx + 2B = x + 14$

$\therefore (A + B)x + (5A + 2B) = x + 14$

$\therefore A + B = 1$ and $5A + 2B = 14$

$A + B = 1$

$\therefore A = 1 - B$ ①

$5A + 2B = 14$ ②

Substitute equation ① into equation ②:

$5(1 - B) + 2B = 14$

$\therefore 5 - 5B + 2B = 14$

$\therefore -3B = 9$

$\therefore B = -3$ ③

Substitute equation ③ into equation ①:

$A = 1 - B = 1 - (-3) = 1 + 3 = 4$

$\therefore A = 4$ and $B = -3$

So now we have:

$\frac{x + 14}{(x + 2)(x + 5)} = \frac{4}{x + 2} + \frac{-3}{x + 5}$

$\therefore \int \frac{x + 14}{(x + 2)(x + 5)}\, dx = \int \left(\frac{4}{x + 2} - \frac{3}{x + 5}\right) dx$

$= 4\int \frac{1}{x + 2}\, dx - 3\int \frac{1}{x + 5}\, dx$

$= 4\, ln\, (x + 2) - 3\, ln\, (x + 5) + c$

4. $\int \frac{3x - 7}{(x^2 - 4x + 3)}\, dx = \int \frac{3x - 7}{(x - 1)(x - 3)}\, dx$

Let $\frac{3x - 7}{(x - 1)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 3}$

$= \frac{A(x - 3) + B(x - 1)}{(x - 1)(x - 3)}$

$\therefore \frac{3x - 7}{(x - 1)(x - 3)} = \frac{Ax - 3A + Bx - B}{(x - 1)(x - 3)}$

$\therefore Ax - 3A + Bx - B = 3x - 7$

$\therefore (A + B)x + (-3A - B) = 3x - 7$

$\therefore A + B = 3$ and $-3A - B = -7$

$A + B = 3$

$\therefore A = 3 - B$ ①

$3A + B = 7$ ②

Substitute equation ① into equation ②:

$3(3 - B) + B = 7$

$9 - 3B + B = 7$

$\therefore -2B = -2$

$\therefore B = 1$ ③

Substitute equation ③ into equation ①:

A = 3 − B = 3 − 1 = 2

∴ A = 2 and B = 1

So now we have:

$$\frac{3x - 7}{(x - 1)(x - 3)} = \frac{2}{x - 1} + \frac{1}{x - 3}$$

$$\therefore \int \frac{3x - 7}{(x - 1)(x - 3)}\, dx = \int\left(\frac{2}{x - 1} + \frac{1}{x - 3}\right) dx$$

$$= 2\int \frac{1}{x - 1}\, dx + \int \frac{1}{x - 3}\, dx$$

$$= 2\, ln\, (x - 1) + ln\, (x - 3) + c$$

5. $\int \frac{5x - 8}{(x^2 - 2x - 8)}\, dx = \int \frac{5x - 8}{(x - 4)(x + 2)}\, dx$

Let $\frac{5x - 8}{(x - 4)(x + 2)} = \frac{A}{x - 4} + \frac{B}{x + 2}$

$= \frac{A(x + 2) + B(x - 4)}{(x - 4)(x + 2)}$

$\therefore \frac{5x - 8}{(x - 4)(x + 2)} = \frac{Ax + 2A + Bx - 4B}{(x - 4)(x + 2)}$

∴ Ax + 2A + Bx − 4B = 5x − 8

∴ (A + B)x + (2A − 4B) = 5x − 8

∴ A + B = 5 and 2A − 4B = −8

A + B = 5

∴ A = 5 − B ①

2A − 4B = −8

∴ A − 2B = −4 ②

Substitute equation ① into equation ②:

5 − B − 2B = −4

∴ −3B = −9

∴ B = 3 ③

Substitute equation ③ into equation ①:

A = 5 − B = 5 − 3 = 2

∴ A = 2 and B = 3

So now we have:

$$\frac{5x - 8}{(x - 4)(x + 2)} = \frac{2}{x - 4} + \frac{3}{x + 2}$$

$$\therefore \int \frac{5x - 8}{(x - 4)(x + 2)}\, dx = \int\left(\frac{2}{x - 4} + \frac{3}{x + 2}\right) dx$$

$$= 2\int \frac{1}{x - 4}\, dx + 3\int \frac{1}{x + 2}\, dx$$

$$= 2\, ln\, (x - 4) + 3\, ln\, (x + 2) + c$$

6. $\int \frac{-4x + 12}{(x^2 + 6x - 7)}\, dx = \int \frac{-4x + 12}{(x + 7)(x - 1)}\, dx$

Let $\frac{-4x + 12}{(x + 7)(x - 1)} = \frac{A}{x + 7} + \frac{B}{x - 1}$

$= \frac{A(x - 1) + B(x + 7)}{(x + 7)(x - 1)}$

$$\therefore \frac{-4x + 12}{(x + 7)(x - 1)} = \frac{Ax - A + Bx + 7B}{(x + 3)(x - 1)}$$

$\therefore Ax - A + Bx + 7B = -4x + 12$

$\therefore (A + B)x + (-A + 7B) = -4x + 12$

$\therefore A + B = -4$ and $-A + 7B = 12$

$A + B = -4$

$\therefore A = -4 - B$ ①

$-A + 7B = 12$ ②

Substitute equation ① into equation ②:

$-(-4 - B) + 7B = 12$

$4 + B + 7B = 12$

$\therefore 8B = 8$

$\therefore B = 1$ ③

Substitute equation ③ into equation ①:

$A = -4 - B$

$= -4 - 1$

$= -5$

$\therefore A = -5$ and $B = 1$

So now we have:

$$\frac{-4x + 12}{(x + 7)(x - 1)} = \frac{-5}{x + 7} + \frac{1}{x - 1}$$

$$\therefore \int \frac{-4x + 12}{(x + 7)(x - 1)}\, dx = \int \left(\frac{-5}{x + 7} + \frac{1}{x - 1}\right) dx$$

$$= -5\int \frac{1}{x + 7}\, dx + \int \frac{1}{x - 1}\, dx$$

$$= -5\, ln\, (x + 7) + ln\, (x - 1) + c$$

Exercise 12.6 (page 198)

1. **a)** Area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times 3 \times 6$

$= 9$ units^2

b) $\int(-2x + 6)\, dx = -\frac{2x^2}{2} + 6x + c = -x^2 + 6x + c$

Area $= \int_0^3 (-2x + 6)\, dx$

$= (-x^2 + 6x)$

$= [-(3)^2 + 6(3)] - [-(0)^2 + 6(0)]$

$= -9 + 18 + 0$

$= 9$ units^2

2. **a)** Estimate the shaded area by counting the squares that are shaded.

Estimate: about 21 units^2

b) $\int(-x^2 + 7x - 6)\, dx = -\frac{x^3}{3} + \frac{7x^2}{2} - 6x + c$

$$\text{Area} = \int_1^6 (-x^2 + 7x - 6)\, dx$$
$$= \left(-\tfrac{x^3}{3} + \tfrac{7x^2}{2} - 6x\right)$$
$$= \left[-\tfrac{6^3}{3} + \tfrac{7(6)^2}{2} - 6(6)\right] - \left[-\tfrac{1^3}{3} + \tfrac{7(1)^2}{2} - 6(1)\right]$$
$$= -72 + 126 - 36 - \left(-\tfrac{1}{3} + \tfrac{7}{2} - 6\right)$$
$$= -72 + 126 - 36 + \tfrac{1}{3} - \tfrac{7}{2} + 6$$
$$= 20.83 \text{ units}^2$$

3. $\int(-x^3 + 4x^2 - 3x)\, dx = -\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2} + c$

First, calculate the area above the x-axis.

$$\text{Area} = \int_1^3 (-x^3 + 4x^2 - 3x)\, dx$$
$$= -\tfrac{x^4}{4} + \tfrac{4x^3}{3} - \tfrac{3x^2}{2}$$
$$= \left[-\tfrac{3^4}{4} + \tfrac{4(3)^3}{3} - \tfrac{3(3)^2}{2}\right] - \left[-\tfrac{1^4}{4} + \tfrac{4(1)^3}{3} - \tfrac{3(1)^2}{2}\right]$$
$$= -\tfrac{81}{4} + 36 - \tfrac{27}{2} - \left(-\tfrac{1}{4} + \tfrac{4}{3} - \tfrac{3}{2}\right)$$
$$= -\tfrac{81}{4} + 36 - \tfrac{27}{2} + \tfrac{1}{4} - \tfrac{4}{3} + \tfrac{3}{2}$$
$$= 2.67 \text{ units}^2$$

Calculate the area below the x-axis.

$$\text{Area} = \left|\int_0^1 (-x^3 + 4x^2 - 3x)\, dx\right|$$
$$= \left|-\tfrac{x^4}{4} + \tfrac{4x^3}{3} - \tfrac{3x^2}{2}\right|$$
$$= \left|\left[-\tfrac{1^4}{4} + \tfrac{4(1)^3}{3} - \tfrac{3(1)^2}{2}\right] - \left[-\tfrac{0^4}{4} + \tfrac{4(0)^3}{3} - \tfrac{3(0)^2}{2}\right]\right|$$
$$= \left|-\tfrac{1}{4} + \tfrac{4}{3} - \tfrac{3}{2} - 0\right|$$
$$= |-0.42|$$
$$= 0.42 \text{ units}^2$$

Total shaded area: $2.67 + 0.42 = 3.09 \text{ units}^2$

4. **a)** $\int(2x + 1)\, dx = \frac{2x^2}{2} + x + c = x^2 + x + c$

$$\text{Area} = \int_1^3 (2x + 1)\, dx = (x^2 + x) = (3^2 + 3) - (1^2 + 1)$$
$$= 12 - 2 = 10 \text{ units}^2$$

b) $\int(x^2 - x - 12)\, dx = \frac{x^3}{3} - \frac{x^2}{2} - 12x + c$

$$\text{Area} = \int_{-3}^4 (x^2 - x - 12)\, dx$$
$$= \tfrac{x^3}{3} - \tfrac{x^2}{2} - 12x$$
$$= \left[\tfrac{4^3}{3} - \tfrac{4^2}{2} - 12(4)\right] - \left[\tfrac{(-3)^3}{3} - \tfrac{(-3)^2}{2} - 12(-3)\right]$$
$$= \tfrac{64}{3} - 8 - 48 - \left(-9 - \tfrac{9}{2} + 36\right)$$
$$= \tfrac{64}{3} - 8 - 48 + 9 + \tfrac{9}{2} - 36$$
$$= -57.17$$

But area must be positive, so area = 57.17 units^2.

c) $\int(x^3 + 8)\,dx = \frac{x^4}{4} + 8x + c$

Area = $\int_{-2}^{0} (x^3 + 8)\,dx$

$= (\frac{x^4}{4} + 8x)$

$= \left[\frac{0^4}{4} + 8(0)\right] - \left[\frac{(-2)^4}{4} + 8(-2)\right]$

$= 0 - (4 - 16)$

$= 12$ units^2

d) $\int 6\,dx = 6x + c$

Area = $\int_{2}^{6} 6\,dx = (6x) = 6(6) - 6(2) = 36 - 12$

$= 24$ units^2

5. The line $y = 6$ is parallel to the x-axis. The area bound by the line $y = 6$, the x-axis and the lines $x = 2$ and $x = 6$ is a rectangle with dimensions 6 units by 4 units. The area of this rectangle is 24 units^2.

Exercise 12.7 (page 201)

1. a) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = 2x + 1$, $a = 1$ and $b = 3$.

$\therefore \int_1^3 (2x + 1)\,dx$

$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$= \left(\frac{3-1}{6}\right)\left[f(1) + 4f\left(\frac{1+3}{2}\right) + f(3)\right]$

$= \frac{1}{3}[f(1) + 4f(2) + f(3)]$

$= \frac{1}{3}[3 + 4(5) + (7)]$

$= \frac{1}{3}(3 + 20 + 7)$

$= 10$ units^2

b) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = -2x + 6$, $a = 0$ and $b = 3$.

$\therefore \int_0^3 (-2x + 6)\,dx$

$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$= \left(\frac{3-0}{6}\right)\left[f(0) + 4f\left(\frac{0+3}{2}\right) + f(3)\right]$

$= \frac{1}{2}[f(0) + 4f(1.5) + f(3)]$

$= \frac{1}{2}[6 + 4(3) + 0]$

$= \frac{1}{2}(6 + 12 + 0)$

$= 9$ units^2

c) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = -x^2 + 7x - 6$, $a = 1$ and $b = 6$.

$\therefore \int_1^6 (-x^2 + 7x - 6)\,dx$

$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$= \left(\frac{6-1}{6}\right)\left[f(1) + 4f\left(\frac{1+6}{2}\right) + f(6)\right]$

$= \frac{5}{6}[f(1) + 4f(3.5) + f(6)]$

$= \frac{5}{6}[0 + 4(-12.25 + 24.5 - 6) + (-36 + 42 - 6)]$

$= \frac{5}{6}(0 + 25 + 0)$

$= \frac{125}{6}$

$= 20.83 \text{ units}^2$

d) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = x^2 - x - 12$, $a = -3$ and $b = 4$.

$\therefore \int_{-3}^4 \left|(x^2 - x - 12)\,dx\right|$

$= \left|\left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$

$= \left|\left(\frac{4-(-3)}{6}\right)\left[f(-3) + 4f\left(\frac{-3+4}{2}\right) + f(4)\right]\right|$

$= \left|\frac{7}{6}\left[f(-3) + 4f\frac{1}{(2)} + f(4)\right]\right|$

$= \left|\frac{7}{6}\left[0 + 4\left(\frac{1}{4} - \frac{1}{2} - 12\right) + 0\right]\right|$

$= \left|\frac{7}{6}(-49)\right|$

$= |-57.17|$

$= 57.17 \text{ units}^2$

e) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = x^3 + 8$, $a = -2$ and $b = 0$.

$\therefore \int_{-2}^0 (x^3 + 8)\,dx$

$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$= \left(\frac{0-(-2)}{6}\right)\left[f(-2) + 4f\left(\frac{-2+0}{2}\right) + f(0)\right]$

$= \frac{1}{3}[f(-2) + 4f(-1) + f(0)]$

$= \frac{1}{3}[0 + 4(7) + 8]$

$= \frac{1}{3}(0 + 28 + 8)$

$= \frac{1}{3}(36)$

$= 12 \text{ units}^2$

f) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = -x^3 + 4x^2 - 3x$.

For the first integral, let $a = 1$ and $b = 3$.

$\therefore \int_1^3 (-x^3 + 4x^2 - 3x)\,dx$

$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$= \left(\frac{3-1}{6}\right)\left[f(1) + 4f\left(\frac{1+3}{2}\right) + f(3)\right]$

$= \frac{2}{6}[f(1) + 4f(2) + f(3)]$

$= \frac{1}{3}[(-1 + 4 - 3) + 4(-8 + 16 - 6) + (-27 + 36 - 9)]$

$= \frac{1}{3}(0 + 4(2) + 0)$

$= \frac{8}{3}$

$= 2.67$ units^2

For the second integral, let $a = 0$ and $b = 1$.

$\therefore \left|\int_1^0 (-x^3 + 4x^2 - 3x)\,dx\right|$

$= \left|\left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$

$= \left|\left(\frac{1-0}{6}\right)\left[f(0) + 4f\left(\frac{0+1}{2}\right) + f(1)\right]\right|$

$= \left|\frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]\right|$

$= \left|\frac{1}{6}\left[0 + 4\left(-\frac{1}{8} + 1 - \frac{3}{2}\right) + 0\right]\right|$

$= \left|\frac{1}{6}\left[4\left(-\frac{1}{8} + 1 - \frac{3}{2}\right) + 0\right]\right|$

$= \left|\frac{1}{6}\left[4\left(-\frac{5}{8}\right)\right]\right|$

$= \left|\frac{1}{6}\left(-\frac{5}{2}\right)\right|$

$= \left|-\frac{5}{12}\right|$

$= |-0.42|$

$= 0.42$ units^2

$\therefore \int_1^3 (-x^3 + 4x^2 - 3x)\,dx + \left|\int_1^0 (-x^3 + 4x^2 - 3x)\,dx\right|$

$= 2.67 + 0.42$

$= 3.09$ units^2

2. **a)** Estimated area: 5 units^2

b) $\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = \frac{1}{2}x^2 - 2$, $a = -2$ and $b = 2$.

$\therefore \left|\int_{-2}^2 \left(\frac{1}{2}x^2 - 2\right)\,dx\right|$

$= \left|\left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$

$= \left|\left(\frac{2-(-2)}{6}\right)\left[f(-2) + 4f\left(\frac{-2+2}{2}\right) + f(2)\right]\right|$
$= \left|\frac{4}{6}[f(-2) + 4f(0) + f(2)]\right|$
$= \left|\frac{2}{3}[0 + 4(-2) + 0)]\right|$
$= \left|\frac{2}{3}(-8)\right|$
$= \left|-\frac{16}{3}\right|$
$= |-5.33|$
$= 5.33 \text{ units}^2$

Exercise 12.8 (page 204)

1. **a)** $v(0) = 0.5(0) = 0$
Initial velocity of the object: 0 m/s

b) $v(6) = 0.5(6) = 3$
So, when $t = 6$ seconds, the velocity of the object is 3 m/s.

c) The area below the line represents the integral of velocity, which is distance. So the shaded area in the diagram represents the distance that the object covered between the fourth and the tenth seconds.

d) $\int\left(\frac{1}{2}t\right) dt = \frac{1}{2}\left(\frac{t^2}{2}\right) + c = \frac{1}{4}t^2 + c$
Area $= \int_4^{10}\left(\frac{1}{4}t^2\right)dt = \left(\frac{1}{4}t^2\right) = \frac{1}{4}(10^2) - \frac{1}{4}(4^2)$
$= 25 - 4 = 21 \text{ m}^2$

2. **a)** $f'(x) = 12(1 - x) = 12 - 12x$
$\therefore f(x) = \int(12 - 12x)dx = 12x - \frac{12x^2}{2} + c = 12x - 6x^2 + c$
Since the football was kicked from the ground, we know that when $t = 0$, $f(x) = 0$.
$\therefore 0 = 12(0) - 6(0)^2 + c$
$\therefore c = 0$
$\therefore f(x) = 12x - 6x^2$

b) The maximum height is found where $f'(x) = 0$.
$\therefore 12 - 12x = 0$
$\therefore x = 1$
$f(x) = 12x - 6x^2$
$\therefore f(1) = 12(1) - 6(1)^2 = 6$
So, the maximum height of the football above the ground is 6 m.

c) When the football reaches the ground again, $f(x) = 0$.

$\therefore 12x - 6x^2 = 0$

$\therefore 2x - x^2 = 0$

$\therefore x(2 - x) = 0$

$\therefore x = 0$ or $x = 2$

So, the football will reach the ground again after two seconds.

3. a) $v = \int a\,dt$

$= \int 6t^2\,dt$

$= \frac{6t^3}{3} + c$

$\therefore v = 2t^3 + c$

When $t = 0$, the object is stationary and $v = 0$, so $c = 0$.

$\therefore v = 2t^3$

b) $s = \int v\,dt = \int 2t^3\,dt = \frac{2t^4}{4} + c$

$\therefore s = \frac{1}{2}t^4 + c$

When $t = 0$, the object is at its starting point, so $s = 0$ and $c = 0$.

$\therefore s = \frac{1}{2}t^4$

c) $s = \frac{1}{2}t^4 = \frac{1}{2}(3^4) = 40.5$ m

So, after three seconds, the object is 40.5 m from its starting point.

4. a) Volume $= \int_0^2 \pi[f(x)]^2\,dx = \int_0^2 \pi x^2\,dx$

$= \pi\frac{x^3}{3}$

$= \pi\frac{2^3}{3} - \left[\pi\left(\frac{0^3}{3}\right)\right]$

$= \pi\frac{8}{3} - 0$

$= 8.38 \text{ units}^2$

b) Volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 2^2 \times 2 = 8.38 \text{ units}^2$

Assess your progress (page 205)

1. Students should use differentiation to check their answers.

a) $\int x^6\,dx = \frac{1}{7}x^7 + c$

b) $\int(5x^4 - \frac{1}{2}x^3 + 4x)\,dx = 5\left(\frac{x^5}{5}\right) - \frac{1}{2}\left(\frac{x^4}{4}\right) + 4\left(\frac{x^2}{2}\right) + c$

$= x^5 - \frac{1}{8}x^4 + 2x^2 + c$

c) $\int(2\cos x - \sin x + 7)\,dx = 2\sin x + \cos x + 7x + c$

d) $\int\left(3\sqrt{x} - \sqrt[3]{x^4} - 1\right) dx$

$= \int(3x^{\frac{1}{2}} - x^{\frac{4}{3}} - 1)\ dx$

$= \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} - x + c$

$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{7}{3}}}{\frac{7}{3}} - x + c$

$= 2x^{\frac{3}{2}} - \frac{3}{7}x^{\frac{7}{3}} - x + c$

e) $\int(x^5 + 10x^4 + \frac{1}{3}x^2)\ dx$

$= \frac{x^6}{6} + \frac{10x^5}{5} + \frac{1}{3}\left(\frac{x^3}{3}\right) + c$

$= \frac{x^6}{6} + 2x^5 + \frac{x^3}{9} + c$

2. a) Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$.

$\int\cos^3 x \times \sin x\ dx = -\int\cos^3 x \times (-\sin x)\ dx$

$= -\int u^3 \frac{du}{dx}\ dx$

$= -\int u^3\ du$

$= -\int u^3\ du$

$= -\frac{u^4}{4} + c$

$= -\frac{\cos^4 x}{4} + c$

b) Let $u = x^2 + 5x$. Then $\frac{du}{dx} = 2x + 5$.

$\int(x^2 + 5x)^7(2x + 5)\ dx = \int u^7 \frac{du}{dx}\ dx$

$= \int u^7\ du$

$= \frac{u^8}{8} + c$

$= \frac{1}{8}(x^2 + 5x)^8 + c$

3. a) Let $\frac{6x}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5}$

$= \frac{A(x-5) + B(x+1)}{(x+1)(x-5)}$

$\therefore \frac{6x}{(x+1)(x-5)} = \frac{Ax - 5A + Bx + B)}{(x+1)(x-5)}$

$\therefore Ax - 5A + Bx + B = 6x$

$\therefore (A + B)x + (-5A + B) = 6x$

$\therefore A + B = 6$ and $-5A + B = 0$

$A + B = 6$

$\therefore A = 6 - B$ ①

$-5A + B = 0$ ②

Substitute equation ① into equation ②:

$-5(6 - B) + B = 0$

$\therefore -30 + 5B + B = 0$

$\therefore 6B = 30$

$\therefore B = 5$ ③

Substitute equation ③ into equation ①:

$A = 6 - B = 6 - 5 = 1$

$\therefore A = 1$ and $B = 5$

So now we have:

$$\frac{6x}{(x+1)(x-5)} = \frac{1}{x+1} + \frac{5}{x-5}$$

$$\therefore \int \frac{6x}{(x+1)(x-5)}\,dx = \int\left(\frac{1}{x+1} + \frac{5}{x-5}\right)dx$$

$$= \int \frac{1}{(x+1)}\,dx + 5\int \frac{1}{x-5}\,dx$$

$$= \ln(x+1) + 5\ln(x-5) + c$$

b) $\int \frac{3x-2}{x^2+x-12}\,dx = \int \frac{3x-2}{(x+4)(x-3)}\,dx$

Let $\frac{3x-2}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$

$$= \frac{A(x-3) + B(x+4)}{(x+4)(x-3)}$$

$$\therefore \frac{3x-2}{(x+4)(x-3)} = \frac{Ax - 3A + Bx + 4B)}{(x+4)(x-3)}$$

$\therefore Ax - 3A + Bx + 4B = 3x - 2$

$\therefore (A + B)x + (-3A + 4B) = 3x - 2$

$\therefore A + B = 3$ and $-3A + 4B = -2$

$A + B = 3$

$\therefore A = 3 - B$ ①

$-3A + 4B = -2$ ②

Substitute equation ① into equation ②:

$-3(3 - B) + 4B = -2$

$-9 + 3B + 4B = -2$

$7B = 7$

$\therefore B = 1$ ③

Substitute equation ③ into equation ①:

$A = 3 - B = 3 - 1 = 2$

$\therefore A = 2$ and $B = 1$

So now we have:

$$\frac{3x-2}{(x+4)(x-3)} = \frac{2}{x+4} + \frac{1}{x-3}$$

$$\therefore \int \frac{3x-2}{(x+4)(x-3)}\,dx = \int\left(\frac{2}{x+4} + \frac{1}{x-3}\right)dx$$

$$= 2\int \frac{1}{x+4}\,dx + \int \frac{1}{x-3}\,dx$$

$$= 2\ln(x+4) + \ln(x-3) + c$$

4. **a)** Estimated area: 11 units^2

b) $\int(-x^2 - 2x + 3)\,dx = -\frac{x^3}{3} - \frac{2x^2}{2} + 3x + c$

$$= -\frac{x^3}{3} - x^2 + 3x + c$$

$$\text{Area} = \int_{-3}^{1}(-x^2 - 2x + 3)\, dx$$
$$= \left(-\frac{x^3}{3} - x^2 + 3x\right)$$
$$= \left[-\frac{1^3}{3} - (1)^2 + 3(1)\right] - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3)\right]$$
$$= -\frac{1}{3} - 1 + 3 - (9 - 9 - 9)$$
$$= -\frac{1}{3} - 1 + 3 + 9$$
$$= 10.67 \text{ units}^2$$

c) $\int_a^b f(x)\, dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

Let $f(x) = -x^2 - 2x + 3$, $a = -3$ and $b = 1$.

$$\therefore \int_{-3}^{1}(-x^2 - 2x + 3)\, dx$$
$$= \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$= \left[\frac{1-(-3)}{6}\right]\left[f(-3) + 4f\left(\frac{-3+1}{2}\right) + f(1)\right]$$
$$= \frac{4}{6}[f(-3) + 4f(-1) + f(1)]$$
$$= \frac{2}{3}[(-9 + 6 + 3) + 4(-1 + 2 + 3) + (-1 - 2 + 3)]$$
$$= \frac{2}{3}(0 + 16 + 0)$$
$$= \frac{32}{3}$$
$$= 10.67 \text{ units}^2$$

5. a) $v = \int a\, dt$
$$= \int 21t^2\, dt$$
$$= \frac{21t^3}{3} + c$$
$$\therefore v = 7t^3 + c$$

When $t = 0$, the velocity of the particle is 10 mm/s, so $c = 10$.

$$\therefore v = 7t^3 + 10$$

b) $s = \int v\, dt$
$$= \int(7t^3 + 10)\, dt$$
$$\therefore s = \frac{7t^4}{4} + 10t + c$$

When $t = 0$, the particle is at its starting point, so $s = 0$ and $c = 0$.

$$\therefore s = \frac{7t^4}{4} + 10t$$

c) $s = \frac{7t^4}{4} + 10t = \frac{7(5)^4}{4} + 10(5) = 1.14 \text{ m}$

So, after five seconds the particle is 1.14 m from its starting point.

TOPIC 13

Revision

Introduction

This final topic is a revision of all of the concepts covered in Mathematics throughout SS1, SS2 and this year.
The material is arranged by theme and follows a logical progression through the curriculum. The revision material for each theme has the following structure:

- A bulleted summary of the main points covered in each topic
- A revision exercise with two parts:
  - Part I consists of multiple-choice questions.
  - Part II consists of questions to which students must give written answers.

Preparation

Display the posters and other material that you prepared and used from SS1 to SS3 in your classroom for your students. Before presenting this topic to your class, give careful thought to how you want to work though this revision material. You should use all the exercises at your own discretion. In some cases, you might choose to only set a few selected questions for your students.

Introducing students to the topic

Explain to your class how you are going to work through the revision material. Certain sections will benefit by a class discussion; you can give other sections for homework and you can ask students to work in pairs or groups for certain sections. By varying your methodology, you will hold your students' interest in the work.

Motivate your students to view this revision topic as an opportunity to prepare for the final examinations. They should use this topic to help them discover which sections of the work they are comfortable with and, more importantly, for which sections of the work they need more practice.

Advise students to not guess answers to multiple-choice questions if they do not know an answer. It would be better for them to ask you for help − this will help you realise which questions you should explain to certain students and which ones should be explained to the whole class.

Answers

Exercise 13.1 (page 216)

1. D	**2.** C	**3.** B	**4.** A	**5.** B	**6.** D
7. C	**8.** B	**9.** A	**10.** D	**11.** A	**12.** D
13. C	**14.** B	**15.** C	**16.** A	**17.** D	**18.** C
19. B	**20.** D	**21.** B	**22.** C	**23.** D	

24. $100010_2 = 1 \times 2^5 + 1 \times 2^1 = 32 + 2 = 34$
$133_4 = 1 \times 4^2 + 3 \times 4^1 + 3 = 16 + 12 + 3 = 31$
$120_5 = 1 \times 5^2 + 2 \times 5^1 = 25 + 10 = 35$
$41_8 = 4 \times 8^1 + 1 = 32 + 1 = 33$
$20_{16} = 2 \times 16^1 = 32$
Ascending order: 133_4, 20_{16}, 41_8, 100010_2, 120_5

25. $32.13_4 = 3 \times 4^1 + 2 \times 4^0 + 1 \times 4^{-1} + 3 \times 4^{-2}$
$= 12 + 2 + \frac{1}{4} + \frac{3}{16}$
$= 14.4375$

26. **a)** $128 = 8 \bmod 12$; $3 + 8 = 11$
The hour hand will point to 11.
b) $-128 = 4 \bmod 12$; $3 + 4 = 7$
The hour hand will point to 7.

27. **a)** 10 m **b)** 350 m **c)** 250 m **d)** 0 m

28. $(2.93 \times 10^{-4}) + (7.87 \times 10^{-3})$
$= (0.293 \times 10^{-3}) + (7.87 \times 10^{-3}) = 8.163 \times 10^{-3}$

29. Descending order: 6.12×10^{-2}, 6.21×10^{-3}, 6.12×10^{-3}, 6.21×10^{-4}, 6.12×10^{-4}

30. $16^x = 8^y$
$\therefore 2^{4x} = 2^{3y}$
$\therefore 4x = 3y$
$\therefore \frac{x}{y} = \frac{3}{4}$

31. $\frac{3^{x+1}}{2^{3-x}} \div \frac{9^{1-x}}{2^{2x+5}} \times \frac{6^{2-3x}}{4}$

$= \frac{3^{x+1}}{2^{3-x}} \times \frac{2^{2x+5}}{9^{1-x}} \times \frac{6^{2-3x}}{4}$

$= \frac{3^{x+1}}{2^{3-x}} \times \frac{2^{2x+5}}{(3^2)^{1-x}} \times \frac{2^{2-3x} \times 3^{2-3x}}{2^2}$

$= 3^{x+1} \times 2^{x-3} \times 2^{2x+5} \times 3^{2x-2} \times 2^{2-3x} \times 3^{2-3x} \times 2^{-2}$

$= 2^{x-3+2x+5+2-3x-2} \times 3^{x+1+2x-2+2-3x}$

$= 2^2 \times 3^1$

$= 12$

32. $\frac{\log \sqrt{25}}{\log \sqrt{125}} + \log 100 - \log 0.1$

$= \frac{\log 5}{\log 5^{\frac{3}{2}}} + \log 10^2 - \log 10^{-1}$

$= \frac{\log 5}{\frac{3}{2}\log 5} + 2\log 10 - (-1)\log 10$

$= \frac{2}{3} + 2 + 1$

$= 3\frac{2}{3}$

33. $6^x = 120$

$\therefore \log 6^x = \log 120$

$\therefore x \log 6 = \log 120$

$\therefore x = \frac{\log 120}{\log 6}$

$= 2.67$

34. $\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{2}} = \frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{10}}$

$= \frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$

$= \frac{\sqrt{20} + 2\sqrt{50}}{10}$

$= \frac{2\sqrt{5} + 10\sqrt{2}}{10}$

$= \frac{\sqrt{5} + 5\sqrt{2}}{5}$

35. $\sqrt{27}(x + 1) = \sqrt{48} + \sqrt{3}$

$\therefore 3\sqrt{3}(x + 1) = 4\sqrt{3} + \sqrt{3}$

$\therefore 3\sqrt{3}(x + 1) = 5\sqrt{3}$

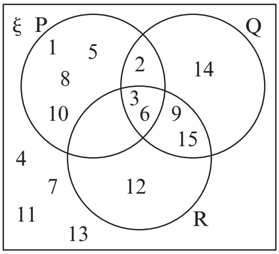
$\therefore 3(x + 1) = 5$

$\therefore 3x + 3 = 5$

$\therefore 3x = 2$

$\therefore x = \frac{2}{3}$

36. a)



b) i) $P \cap Q = \{2, 3, 6\}$

ii) $Q \cup R = \{2, 3, 6, 9, 12, 14, 15\}$

iii) $(P \cup Q \cup R)'$
$= \{4, 7, 11, 13\}$

iv) $(P \cap Q \cap R)$
$= \{3, 6\}$

37. a) i) The students in SS3 are not the only students at the school.

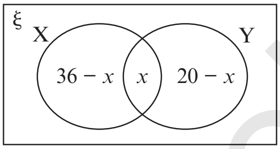
ii) Some students in SS3 play a musical instrument.

iii) All the students in SS3 are girls.

b) i) $G \cap M \neq \{\}$ **ii)** $S \cap G' \neq \{\}$

iii) $S \cap M = \{\}$

38.



Let $n(X \cap Y) = x$.

$\therefore 36 - x + x + 20 - x = 44$

$\therefore -x = 44 - 56$

$= -12$

$\therefore x = 12$

$\therefore n(X \cap Y) = 12$

39. a) $T_n = a + (n - 1)d$

$T_5 = a + (5 - 1)d$

$= a + 4d$

$\therefore a + 4d = 40$ ①

$T_{15} = a + (15 - 1)d$

$= a + 14d$

$\therefore a + 14d = 10$ ②

Subtract equation ① from equation ②:

$\therefore 10d = -30$

$\therefore d = -3$

b) $a + 4d = 40$

$\therefore a + 4(-3) = 40$

$\therefore a = 52$

c) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{40} = \frac{40}{2}[2(52) + (40 - 1)(-3)] = 20(-13) = -260$

40. $S_\infty = \frac{a}{1 - r}$

$\therefore \frac{a}{1 - (-0.1)} = \frac{4\,500}{99}$

$\therefore \frac{a}{1.1} = \frac{4\,500}{99}$

$\therefore a = \frac{4\,500}{99} \times 1.1$

$= 50$

$T_1 = 50$; $T_2 = 50 \times -0.1 = -5$ and $T_3 = -5 \times -0.1 = 0.5$

The first three terms: 50, −5 and 0.5

41. a) $|\mathbf{A}| = -4 - (-6) = 2$

b) $\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1.5 & -0.5 \end{pmatrix}$

c) $\mathbf{A} \times \mathbf{A}^{-1} = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ 1.5 & -0.5 \end{pmatrix}$

$= \begin{pmatrix} -2 + 3 & 1 - 1 \\ -6 + 6 & 3 - 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

42. $2a - 5 = -5$

$\therefore a = 0$

$2 - 5b = -13$

$\therefore -5b = -15$

$\therefore b = 3$

$c = -1 + 3b = -1 + 3(3) = 8$

$d = -1a + 3 = -1(0) + 3 = 3$

43. a) 20% of ₦350 000 = ₦70 000

b) Total price = ₦70 000 + 24 × ₦14 350 = ₦414 400

c) Interest charged = ₦414 400 − ₦350 000 = ₦64 400

Amount borrowed = ₦350 000 − ₦70 000 = ₦280 000

$\frac{64\,400}{280\,000} = 0.23$

Interest rate: 23%

44. a) $A = P(1 + i)^n = 225\,000(1 + 0.06)^3$ = ₦270 000 (to the nearest ₦10 000)

b) $A = P(1 + i)^n$

$\therefore P(1 + 0.0875)^3 = 270\,000$

$\therefore P = \frac{270\,000}{(1 + 0.0875)^3}$

= ₦210 000 (to the nearest ₦10 000)

45. **a)** $h = \frac{V}{\pi r^2}$ **b)** h varies directly with V and inversely with the square of r.

46. **a)** $y = a + bx^2$

b) $8 = a + b(1)^2$

$\therefore a + b = 8$ ①

$17 = a + b(2)^2$

$\therefore a + 4b = 17$ ②

② − ①:

$3b = 9$

$\therefore b = 3$ ③

Substitute equation ③ into equation ①:

$a + 3 = 8$

$\therefore a = 5$

$y = 5 + 3x^2$

When $x = 5$: $y = 5 + 3(5)^2 = 5 + 75 = 80$

Exercise 13.2 (page 224)

1.	C	**2.**	A	**3.**	D	**4.**	C	**5.**	B	**6.**	C
7.	A	**8.**	B	**9.**	D	**10.**	A	**11.**	C	**12.**	B

13. $x^2 - ay - y^2 + ax = x^2 - y^2 + ax - ay$

$= (x + y)(x - y) + a(x - y)$

$= (x - y)(x + y + a)$

14. **a)** The expression is undefined if either denominator is zero.

$6x^2 - 54 = 0$

$\therefore x^2 - 9 = 0$

$\therefore (x + 3)(x - 3) = 0$

$\therefore x = -3$ or $x = 3$

$25 - x^2 = 0$

$\therefore (5 + x)(5 - x) = 0$

$\therefore x = -5$ or $x = 5$

The expression is undefined if $x = -3$ or $x = 3$ or $x = -5$ or $x = 5$.

b) $\frac{x^2 - 2x - 15}{6x^2 - 54} \div \frac{25 - x^2}{3x^2 - 3x - 18} = \frac{x^2 - 2x - 15}{6x^2 - 54} \times \frac{3x^2 - 3x - 18}{25 - x^2}$

$= \frac{(x - 5)(x + 3)}{6(x^2 - 9)} \times \frac{3(x^2 - x - 6)}{(5 + x)(5 - x)}$

$= \frac{(x - 5)(x + 3)}{6(x + 3)(x - 3)} \times \frac{3(x - 3)(x + 2)}{(5 + x)(5 - x)}$

$= -\frac{x + 2}{2(x + 5)}$

15. a) $-6(5)^2 + k(5) + 135 = 0$

$\therefore -150 + 5k + 135 = 0$

$\therefore 5k = 15$

$\therefore k = 3$

b) $-6x^2 + 3x + 135 = 0$

$\therefore -2x^2 + x + 45 = 0$

$\therefore 2x^2 - x - 45 = 0$

$\therefore (2x + 9)(x - 5) = 0$

$\therefore x = -\frac{9}{2}$ or $x = 5$

The other root is $-\frac{9}{2}$.

16. a) $3x^2 + 8x - 3 = 0$

$\therefore x^2 + \frac{8}{3}x - 1 = 0$

$\therefore x^2 + \frac{8}{3}x = 1$

$\therefore x^2 + \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = 1 + \left(\frac{4}{3}\right)^2$

$\therefore \left(x + \frac{4}{3}\right)^2 = 1 + \frac{16}{9}$

$= \frac{25}{9}$

$\therefore x + \frac{4}{3} = \pm\sqrt{\frac{25}{9}}$

$\therefore x = -\frac{4}{3} \pm \frac{5}{3}$

$\therefore x = \frac{1}{3}$ or $x = -\frac{9}{3}$

$\therefore x = \frac{1}{3}$ or $x = -3$

b) $3x^2 + 8x - 3 = 0$

$\therefore (3x - 1)(x + 3) = 0$

$\therefore x = \frac{1}{3}$ or $x = -3$

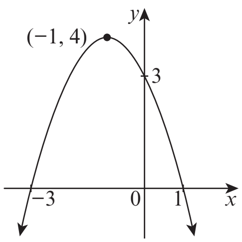
17. a) $y = 3$

b) $x = 1$ or $x = -3$

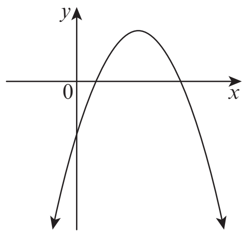
c) $x = -1$

d) The arms of the parabola will go down.

e)



18.



Important features of this graph:

- $a < 0$, so the arms of the parabola point downwards.
- $c < 0$, so the y-intercept is negative.
- $-\frac{b}{2a} > 0$, so the turning point has a positive x-value.
- The function has two real unequal roots, so there are two different x-intercepts.

19. $y = 2x^2 + 5x + 5$ ①

$y = -3(x + 1)$

$\therefore y = -3x - 3$ ②

Substitute equation ① into equation ②:

$2x^2 + 5x + 5 = -3x - 3$

$\therefore 2x^2 + 8x + 8 = 0$

$\therefore x^2 + 4x + 4 = 0$

$\therefore (x + 2)(x + 2) = 0$

$\therefore x = -2$ ③

Substitute equation ③ into equation ②:

$y = -3(-2) - 3$

$\therefore y = 3$

20. $x^2 + y^2 = 9$ ①

$y = \frac{1}{3}x - 1$ ②

Substitute equation ② into equation ①:

$x^2 + \left(\frac{1}{3}x - 1\right)^2 = 9$

$\therefore x^2 + \frac{1}{9}x^2 - \frac{2}{3}x + 1 = 9$

$\therefore 9x^2 + x^2 - 6x + 9 = 81$

$\therefore 10x^2 - 6x - 72 = 0$

$\therefore 5x^2 - 3x - 36 = 0$

$\therefore (5x + 12)(x - 3) = 0$

$\therefore x = -\frac{12}{5}$ or $x = 3$ ③

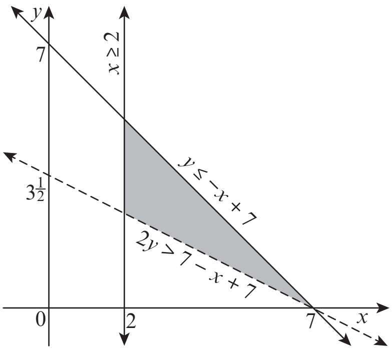
Substitute equation ③ into equation ②:

If $x = -\frac{12}{5}$: $y = \frac{1}{3}\left(-\frac{12}{5}\right) - 1 = -\frac{4}{5} - 1 = -\frac{9}{5}$

If $x = 3$: $y = \frac{1}{3}(3) - 1 = 0$

The coordinates of A are $\left(-\frac{12}{5}; -\frac{9}{5}\right)$ and the coordinates of B are (3; 0).

21.



22. $x \geq 0$ and $y \geq 0$
$100 \leq x + y \leq 400$
$x \geq 2y$

23. **a)** Oluwa is not a baby.
b) Oluwa is not crying.
c) Oluwa is a baby and Oluwa is sleeping.
d) Oluwa is a baby or Oluwa is sleeping.
e) Oluwa is sleeping.
f) If Oluwa is a baby, then Oluwa is sleeping.
g) Oluwa is sleeping if and only if Oluwa is a baby.
h) If Oluwa is a baby, then Oluwa is not sleeping.

24.

P	Q	~Q	P ∧ ~Q	P ∧ Q	(P ∧ ~Q) ∨ (P ∧ Q)
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	F	F	F
F	F	T	F	F	F

Exercise 13.3 (page 236)

1. A	**2.** C	**3.** B	**4.** D	**5.** A	**6.** A
7. C	**8.** D	**9.** B	**10.** A	**11.** B	**12.** C
13. D	**14.** B	**15.** A	**16.** B	**17.** C	**18.** D

19. **a)** A parallelogram consists of two congruent triangles. If only two sides of a triangle are given, many different triangles can be drawn.

b) The teacher must give either the third side of the triangle (which would be a diagonal of the parallelogram); alternatively she should give the size of the angle between the two given sides.

20. $X\hat{Y}Z = 60°$

21. $P\hat{U}T = 90°$ (Corresponding angles; QR ∥ UT)
$\therefore x = 90° + 55° = 145°$ (Exterior angle of △PUT)

22. **a)** $\frac{BE}{BC} = \frac{BD}{AB} = \frac{3}{4}$ (DE ∥ AC)

b) area △BDE : area △ABC
$= \frac{1}{2} \times BE \times BD : \frac{1}{2} \times BC \times AB$
$= \frac{1}{2} \times \frac{3}{4} \times BC \times \frac{3}{4} \times AB : \frac{1}{2} \times BC \times AB$
$= \frac{3}{4} \times \frac{3}{4} : 1$
$= \frac{9}{16} : 1$
$= 9 : 16$

c) Let the area of △BDE be $9x$ cm^2.
Then, the area of △ABC is $16x$ cm^2.
$16x = 9x + 7$
$\therefore 7x = 7$
$\therefore x = 1$ cm^2
So, the area of △BDE is 9 cm^2.

23. Arc length $= \frac{90}{360} \times 2 \times \pi \times 21 = 33$ mm
Chord AB $= \sqrt{21^2 + 21^2}$
$= 29.70$ mm (Pythagoras' theorem)
33 mm + 29.7 mm = 62.7 mm
Perimeter of the segment: 62.7 mm

24. Area of sector $= \frac{15}{360} \times \pi r^2$
$\therefore \frac{15}{360} \times 3.14 \times r^2 = 13$
$\therefore r^2 = \frac{13 \times 24}{3.14}$
$\therefore r = \sqrt{\frac{13 \times 24}{3.14}}$
$= 9.97$ cm

25. $P\hat{S}R = x$ (Angle at centre = twice angle at circumference)
$P\hat{Q}R + P\hat{S}R = 180°$ (Opposite angles in a cyclic quadrilateral)
$\therefore 2x + x = 180°$
$\therefore 3x = 180°$
$\therefore x = 60°$

26. OA = OB = 33 mm (Radii)

$\hat{OAT} = 90°$ (Tangent perpendicular to radius)

$\therefore OT^2 = OA^2 + AT^2$

$= 33^2 + 70^2$

$= 5\ 989$ (Pythagoras' theorem)

$\therefore OT = \sqrt{5\ 989} = 77.39$ mm

$\therefore BT = 77.39 \text{ mm} - 33 \text{ mm}$

$= 44$ mm (correct to two significant figures)

27. TSA of a sphere $= 4\pi r^2$

Let r be the radius of the smaller sphere and R be the radius of the larger sphere.

$\frac{4\pi r^2}{4\pi R^2} = \frac{25}{36}$

$\therefore \frac{r^2}{R^2} = \frac{25}{36}$

$\therefore \frac{r^2}{18^2} = \frac{25}{36}$

$\therefore \frac{r^2}{324} = \frac{25}{36}$

$\therefore r^2 = \frac{25 \times 324}{36}$

$= 225$

$\therefore r = 15$ cm

$\therefore \text{Volume} = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times 15^3 = 14\ 143 \text{ cm}^3$

28. a) TSA $= \pi(R + r)l + \pi(R^2 + r^2)$, where l is the slant height of the frustum

$l = \sqrt{24^2 + 7^2} = 25$ cm (Pythagoras' theorem)

TSA $= \pi(R + r)l + \pi(R^2 + r^2)$

$= \frac{22}{7}(14 + 7)(25) + \frac{22}{7}(14^2 + 7^2)$

$= 1\ 650.01 + 770$

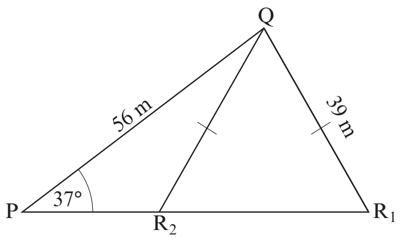
$= 2\ 420 \text{ cm}^2$

b) Volume $= \frac{\pi h}{3}(R^2 + Rr + r^2)$

$= \frac{22}{7} \times \frac{24}{3} \times (14^2 + 14 \times 7 + 7^2)$

$= 8\ 624 \text{ cm}^3$

29.



$\frac{\sin \hat{R}}{r} = \frac{\sin \hat{P}}{p}$ (sine rule)

$\therefore \frac{\sin \hat{R}}{56} = \frac{\sin 37°}{39}$

$\therefore \sin \hat{R} = \frac{56 \sin 37°}{39}$

$\therefore \hat{R} = 59.8°$

Or, $\hat{R} = 180° - 59.8° = 120.2°$

First solution, using $\hat{R} = 59.8°$:

$\hat{Q} = 180° - 37° - 59.8° = 83.2°$ (Sum of angles of a triangle)

$\frac{q}{\sin \hat{Q}} = \frac{p}{\sin \hat{P}}$ (sine rule)

$\therefore \frac{PR}{\sin 83.2°} = \frac{39}{\sin 37°}$

$\therefore PR = \frac{39 \sin 83.2°}{\sin 37°}$

$= 64.35$ m

Second solution, using $\hat{R} = 120.2°$:

$\hat{Q} = 180° - 37° - 120.2° = 22.8°$ (Sum of angles of a triangle)

$\frac{q}{\sin \hat{Q}} = \frac{p}{\sin \hat{P}}$ (sine rule)

$\therefore \frac{PR}{\sin 22.8°} = \frac{39}{\sin 37°}$

$\therefore PR = \frac{39 \sin 22.8°}{\sin 37°}$

$= 25.11$ m

30. a) The length of one side of the hexagon is x units. The hexagon consists of six equilateral triangles with sides of x units each. From the special 60°/30° triangle, we know that the vertical height of each triangle will be $\frac{x}{2} \times \sqrt{3}$ units.

Area of one triangle $= \frac{1}{2} \times x \times \frac{x}{2} \times \sqrt{3}$ units^2

Area of the hexagon $= 6 \times \frac{1}{2} \times x \times \frac{x}{2} \times \sqrt{3}$

$= \frac{3}{2} \times \sqrt{3} \times x^2$

$= \frac{\sqrt{27}\, x^2}{2}$ units^2

b) $x = \frac{42}{6}$ cm $= 7$ cm

$\therefore A = \frac{\sqrt{27}\, 7^2}{2}$ units^2 $= 127.31$ units^2

31. $B\hat{I}R = 30.5°$ (Alternate angles; BD || RI)

$\frac{BR}{BI} = \sin B\hat{I}R$

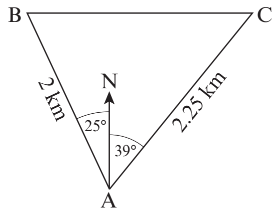
$\therefore \frac{BR}{4.25} = \sin 30.5°$

$\therefore BR = 4.25 \sin 30.5°$

$= 2.16$ m

32. a) Distance = speed × time

$\therefore$ AB = $\frac{1}{2} \times 4 = 2$ km and AC = $\frac{1}{2} \times 4.5 = 2.25$ km

b) 

c) $B\hat{A}C = 25° + 39° = 64°$

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A} \quad \text{(cosine rule)}$$

$$\therefore BC^2 = 2^2 + 2.25^2 - 2(2)(2.25) \cos 64°$$

$$\therefore BC = \sqrt{2^2 + 2.25^2 - 2(2)(2.25) \cos 64°}$$

$$= 2.26 \text{ km}$$

Distance between the hikers: 2.26 km

33. Angle subtended by arc AB at the centre of the earth

= 74° + 3° = 77°

$\therefore$ Arc AB = $2\pi R \times \frac{\theta}{360°} = 2 \times 3.142 \times 6\,400 \times \frac{77°}{360°}$

= 8 602 km (to the nearest kilometre)

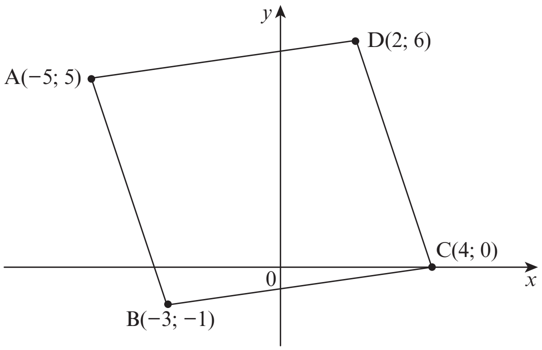
34. The difference between the coordinates of longitude

= 151° − 0.0° = 151°

Time difference = $\frac{151°}{15°}$ hours = 10 hours (to the nearest 30 minutes)

Sydney is further east than Greenwich, so it is 10 hours ahead of Greenwich.

So, the time in Sydney is 1 p.m.

35. a) 

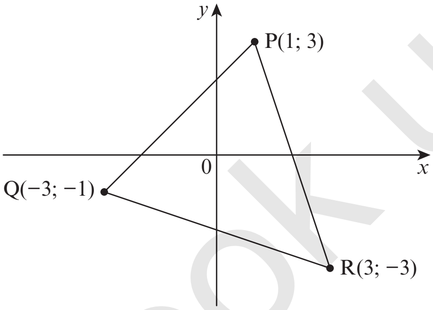
b) $\text{Midpoint}_{AC} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-5 + 4}{2}; \frac{5 + 0}{2}\right) = \left(-\frac{1}{2}; \frac{5}{2}\right)$

$\text{Midpoint}_{BD} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{-3 + 2}{2}; \frac{-1 + 6}{2}\right) = \left(-\frac{1}{2}; \frac{5}{2}\right)$

The midpoints of AC and BD have the same coordinates, so AC and BD bisect each other.

c) ABCD is a parallelogram because the diagonals of ABCD bisect each other. ABCD could also be a rectangle, a rhombus or a square.
If ABCD is a rectangle or a square,
then $m_{AB} \times m_{BC} = -1$.
$m_{AB} \times m_{BC} = -3 \times \frac{1}{7} \neq -1$.
If ABCD is a rhombus, then AB = BC.
$AB = \sqrt{6^2 + 2^2} = \sqrt{40}$; $BC = \sqrt{7^2 + 1^2} = \sqrt{50}$
ABCD is not a rhombus; so, it is a parallelogram.

36. a)



b) $M = \text{Midpoint}_{PQ} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{1 - 3}{2}; \frac{3 - 1}{2}\right) = (-1; 1)$
$N = \text{Midpoint}_{PR} = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right) = \left(\frac{1 + 3}{2}; \frac{3 - 3}{2}\right) = (2; 0)$

c) $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[2 - (-1)]^2 + (0 - 1)^2} = \sqrt{3^2 + (-1)^2}$
$= \sqrt{9 + 1} = \sqrt{10}$ units

$QR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{[3 - (-3)]^2 + [-3 - (-1)]^2} = \sqrt{6^2 + (-2)^2}$
$= \sqrt{36 + 4} = \sqrt{40}$ units $= 2\sqrt{10}$ units
$\therefore MN = \frac{1}{2}QR$

d) $m_{MN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 1}{2 - (-1)} = -\frac{1}{3}$ and $m_{QR} = \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{-3 - (-1)}{3 - (-3)} = \frac{-2}{6} = -\frac{1}{3}$
$\therefore MN \parallel QR$

e) $PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-3 - 3)^2 + (3 - 1)^2}$
$= \sqrt{36 + 4} = \sqrt{40}$ units = QR

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1 - 3)^2 + (-3 - 1)^2}$

$= \sqrt{16 + 16} = \sqrt{32}$ units

So, △PQR is an acute-angled isosceles triangle.

Exercise 13.4 (page 244)

1. C	**2.** B	**3.** C	**4.** A
5. C	**6.** D	**7.** D	**8.** D

9. **a)** Mean $= \frac{-30}{15} = -2$ **b)** Median $= -4$
c) Mode $= 8$ **d)** Range $= 14 - (-14) = 28$
e) $Q_1 = -11$ **f)** $Q_2 = -4$
g) $Q_3 = 8$

10. **a)** $\bar{x} = \frac{115}{5} = 23$ kg

b)

Mass (x)	Mean (x)	Deviation ($x - \bar{x}$)	Square of deviation $(x - \bar{x})^2$
21	23	–2	4
4	23	–19	361
39	23	16	256
19	23	–4	16
32	23	9	81

$\Sigma(x - \bar{x}) = 718$

$\therefore \sigma^2 = \frac{718}{5} = 143.6$

c) $\sigma = \sqrt{143.6} = 11.98$ km

d) $\bar{x} - \sigma = 23 - 11.98 = 11.02$ kg

$\bar{x} + \sigma = 23 + 11.98 = 34.98$ kg

So, 4 kg and 39 kg do not fall within one standard deviation from the mean.

11. **a)** There are 100 pages in her album.
b) **i)** $\frac{410}{100} = 4.1$ photographs per page
ii) 4; the 50th data value corresponds with four photographs per page
iii) Four photographs per page
iv) $8 - 1 = 7$ photographs per page

12. **a)** 70 students were interviewed.
b) The modal class is (20, 30] hours.
c) Estimated mean $= \frac{1\,850}{70} = 26.428\ldots\ h \approx 26$ h 30 min.

13. **a)** Pie chart

b) Modal class: 30 °C − 34 °C (this is the biggest sector)

c) 60° is exactly one sixth of 360°, so the number of days in the year must be divisible by 6.
$\frac{365}{6} = 60\frac{5}{6}$ and $\frac{366}{6} = 61$, so it must have been a leap year.

14. **a)** **i)** Approximately ₦34 000

ii) Approximately ₦47 000

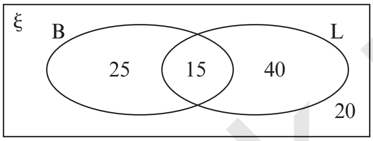
iii) Approximately ₦63 000

iv) Approximately ₦41 000

b) P_{97}

c) 137 families

15. **a)**



b) 100 tourists

16. **a)** $\frac{120}{240} = \frac{1}{2}$ **b)** $\frac{140}{240} = \frac{7}{12}$

17. **a)**

		Yellow spinner					
		4	**5**	**6**	**7**	**8**	**9**
Green spinner	**1**	14	15	16	17	18	19
	2	24	25	26	27	28	29
	3	34	35	36	37	38	39
	4	44	45	46	47	48	49
	5	54	55	56	57	58	59

b) **i)** $\frac{5}{30} = \frac{1}{6}$ **ii)** $\frac{6}{30} = \frac{1}{5}$

iii) $\frac{2}{30} = \frac{1}{15}$ **iv)** 0

Exercise 13.5 (page 251)

1. C **2.** D **3.** B **4.** A

5. $f(x) = 2x^3 - 5x^2 + 4$
$\therefore f(x + h)$
$= 2(x + h)^3 - 5(x + h)^2 + 4$

$= 2(x + h)(x + h)^2 - 5(x^2 + 2xh + h^2) + 4$
$= 2(x + h)(x^2 + 2xh + h^2) - 5x^2 - 10xh - 5h^2 + 4$
$= 2(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) - 5x^2 - 10xh - 5h^2 + 4$
$= 2x^3 + 6x^2h + 6xh^2 + 2h^3 - 5x^2 - 10xh - 5h^2 + 4$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 5x^2 - 10xh - 5h^2 + 4 - (2x^3 - 5x^2 + 4)}{h}$
$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3 - 10xh - 5h^2}{h}$
$= \lim_{h \to 0} (6x^2 + 6xh + 2h^2 - 10x - 5h)$
$= 6x^2 - 10x$
$\therefore f'(x) = 6x^2 - 10x$

6. **a)** $f(x) = x(x + 2)(x - 4) = x(x^2 - 2x - 8)$
$= x^3 - 2x^2 - 8x$
$\therefore f'(x) = 3x^2 - 4x - 8$
If $x = -4$, then $f'(-4) = 56$.
The equation of the tangent is $y = mx + c$.
So, $y = 56x + c$
$f(-4) = -64$, so the tangent goes through the point $(-4; -64)$.
Substitute $(-4; -64)$ into the equation of the tangent:
$y = 56x + c$
$\therefore -64 = 56(-4) + c$
$\therefore c = 160$
Equation of the tangent to $f(x)$ that passes through $(-4; -64)$ is $y = 56x + 160$.

b) $f'(x) = 3x^2 - 4x - 8$
If $x = 0$, then $f'(0) = -8$.
The equation of the tangent is $y = mx + c$.
So, $y = -8x + c$.
If $x = 0$, then $f(0) = 0$; so, the tangent goes through $(0; 0)$.
Substitute $(0; 0)$ into the equation of the tangent:
$y = -8x + c$
$\therefore 0 = 0 + c$
$\therefore c = 0$
Equation of the tangent to $f(x)$ that passes through $(0; 0)$ is $y = -8x$.

7. $\int \frac{2(x-7)}{x^2+x-6}\,dx = \int \frac{2x-14}{(x+3)(x-2)}\,dx$

Let $\frac{2x-14}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

$= \frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$

$\therefore \frac{2x-14}{(x+3)(x-2)} = \frac{Ax-2A+Bx+3B}{(x+3)(x-2)}$

$\therefore Ax - 2A + Bx + 3B = 2x - 14$

$\therefore (A+B)x + (-2A+3B) = 2x - 14$

$\therefore A + B = 2$ and $-2A + 3B = -14$

$A + B = 2$

$\therefore A = 2 - B$ ①

$-2A + 3B = -14$ ②

Substitute equation ① into equation ②:

$-2(2-B) + 3B = -14$

$-4 + 2B + 3B = -14$

$5B = -10$

$\therefore B = -2$ ③

Substitute equation ③ into equation ①:

$A = 2 - B = 2 - (-2) = 4$

$\therefore A = 4$ and $B = -2$

So, now we have:

$\frac{2x-14}{(x+3)(x-2)} = \frac{4}{x+3} - \frac{2}{x-2}$

$\therefore \int \frac{2x-14}{(x+3)(x-2)}\,dx = \int\left(\frac{4}{x+3} - \frac{2}{x-2}\right)dx$

$= 4\int \frac{1}{x+3}\,dx - 2\int \frac{1}{x-2}\,dx$

$= 4\ ln\ (x+3) - 2\ ln\ (x-2) + c$

8. $\int(-x^2 + 8x - 12)\,dx = -\frac{x^3}{3} + \frac{8x^2}{2} - 12x + c$

$= -\frac{x^3}{3} + 4x^2 - 12x + c$

Area $= \int_2^6(-x^2 + 8x - 12)\,dx$

$= -\frac{x^3}{3} + 4x^2 - 12x$

$= \left[-\frac{6^3}{3} + 4(6)^2 - 12(6)\right] - \left[-\frac{2^3}{3} + 4(2)^2 - 12(2)\right]$

$= -\frac{216}{3} + 144 - 72 + \frac{8}{3} - 16 + 24$

$= 10.\dot{6}$

$= 10.67$ units^2