Excellence in
Mathematics

SENIOR SECONDARY
2
TEACHER'S GUIDE

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Introduction

The purpose of the curriculum

The main objectives of the Mathematics curriculum are to prepare the students to:

- acquire the mathematical literacy necessary to functioning in an information age
- cultivate the understanding and application of mathematical concepts and skills necessary to thrive in the ever-changing technological world
- develop the essential skills of problem solving, communication, reasoning and connection within the study of Mathematics
- take advantage of the numerous career opportunities provided by Mathematics
- prepare for further studies in Mathematics and other related fields.

The role of the teacher

One of the principal duties of a Mathematics teacher is to prepare and present good lessons to students. In order to do this, the teacher needs to:

- be as well informed as possible on the scheme of work of the subject
- know the aims and objective of each topic
- select appropriate content material
- decide on the best methods of presentation such as group work, worksheets, question–answer sessions, debate, etc.
- keep informed about social and environmental issues and other current news in Nigeria and the rest of the world
- through innovative teaching approaches, encourage learning that will promote creativity and critical thinking.

To be effective in presentation, the teacher should prepare a written or typed plan for each lesson. This must include aims, objectives, resources, time frames, content for the lesson, activities, homework, assessment, and ideas for additional worksheets to cater for students requiring extension or learning support (remediation).

Prepare each topic in advance. It is your responsibility as a Mathematics teacher to involve your students actively in the learning process. It is a proven fact that students learn far more by *doing* than by *listening*.

Mathematics involves being curious and asking questions. Wherever possible ask questions to engage the students, to encourage independent thought processes, and to develop problem-solving skills. Start your lessons by asking the students to write down answers to questions related to your lesson (approximately five). This will settle them into the lesson.

You can use different types of questions in your lessons:

- **diagnostic** questions enabling you to determine prior knowledge on the topic
- for **consolidation** of challenging concepts during the lesson
- for **stimulation** of interest in the subject
- for **concluding** the lesson.

Concluding questions will assist you in finding out whether students have understood the concepts and terminology of the lesson. It will also highlight any areas that they need to revise at home or for you to revisit in the next lesson.

It is best to ensure that you do not appear to have favourites in the class, so devise a system to ask questions fairly, but be careful not to embarrass weak students if they cannot answer.

How to use the scheme of work

A scheme of work is defined as the part of the curriculum that a teacher will be required to teach in any particular subject. Its primary function is to provide an outline of the subject matter and its content, and to indicate how much work a student should cover in any particular class. A scheme of work allows teachers to clarify their thinking about a subject, and to plan and develop particular curriculum experiences that they believe may require more time and attention when preparing lessons. The criteria all teachers should bear in mind when planning a scheme of work are continuity in learning and progression of experience. You can add your own notes to the scheme of work provided on pages vi to viii.

The scheme of work is sequential. The sequence of the scheme of work is aligned with the Student's Book. Do not be tempted to jump around. Rather spend time carefully planning the term to ensure that you adhere to the scheme of work.

The year is divided into three terms, and each term is divided into 13 weeks. There are six topics covered in each term. The end of term allows time for revision and an examination. This time frame may vary depending on the planning of your particular school.

Your management of the class will have an enormous influence on your ability to adhere to the time frames. Focus on effective discipline strategies. You will have fewer discipline issues if you are: punctual, well prepared, follow a plan (write this on the board at the start of the lesson), keep your word (don't make empty threats) and consistently adhere to rules.

A teacher of Mathematics is a professional instructor who facilitates, promotes and influences students to achieve the outcomes of the scheme of work. It is the wish of the authors that the students will, at the end of each course in the series, attain a level of Mathematics proficiency that will equip them for future studies in this field.

Scheme of work

Term 1

Topic	Lesson objectives	Student's Book pages
1. Revision of SS1 work, focusing on logarithms	Students should be able to: • draw on prior knowledge of mathematics learnt SS1	1–21
2. Logarithms	Students should be able to: • compare logarithms with standard form • calculate logarithms of numbers greater than 1 • calculate logarithms of numbers less than 1 • solve problems involving multiplication, division, powers and roots • solve simple logarithmic equations • check accuracy of results using logarithm tables and calculators	25–40
3. Approximations	Students should be able to: • approximate numbers to nearest 10, 100, 1 000, million, billion and trillion • determine accuracy of results using logarithm tables and calculators • calculate percentage error • apply approximation to everyday life	41–52
4. Sequences and series	Students should be able to: • identify types of sequence • for arithmetic progressions, find first term, common difference, nth term, arithmetic mean(s) and sum • for geometric progressions, find first term, common ratio, nth term, geometric mean(s), sum and sum to infinity • apply arithmetic and geometric progressions in real life situations	53–75
5. Quadratic equations	Students should be able to: • revise factorising methods • factorise perfect squares • complete the square • derive and apply quadratic formula • form a quadratic equation from sum and product of given roots • solve quadratic equations in word problems	76–89
6. Simultaneous equations	Students should be able to: • solve simultaneous linear equations by elimination and substitution • draw graphs of linear and quadratic functions • solve simultaneous linear and quadratic equations by algebraic and graphical methods • solve word problems that involve simultaneous equations	90–109

Term 2

Topic	Lesson objectives	Student's Book pages
7. Gradient of a curve	Students should be able to: • determine x and y intercept of any linear graph • find gradient of a straight line • draw graph of a linear equation using gradient method • use graph to determine gradient between two points of a curve • draw tangents to a curve at a given point	110–123
8. Linear inequalities	Students should be able to: • solve simple and compound linear inequalities in one variable • solve linear inequalities in two variables • solve simultaneous linear inequalities and illustrate on Cartesian plane • identify constraints and formulate inequalities from given information about real life problems • find coordinates of vertices of feasible region and identify feasible region • formulate objective functions • solve linear programming problems	124–145
9. Algebraic fractions	Students should be able to: • simplify algebraic fractions • use operations with algebraic fractions • solve equations involving algebraic fractions • substitute into fractions • determine values that make a fraction undefined	146–158
10. Logical reasoning	Students should be able to: • revise basic components of logical reasoning • define converse, inverse and contrapositive of a conditional statement • learn about equivalent statements • apply contrapositives and inverses in proving theories	159–169
11. Circle geometry: Chord properties	Students should be able to: • apply theorem of Pythagoras • describe chord properties of a circle • apply chord properties to solve problems	170–184
12. Circle geometry: Angle properties	Students should be able to: • describe angle properties of a circle • prove theorems involving angle properties • describe properties of cyclic quadrilaterals • apply properties of cyclic quadrilaterals to solve problems	185–199

Term 3

Topic	Lesson objectives	Student's Book pages
13. Circle geometry: Tangents	Students should be able to: • describe properties of tangents to a circle • prove tangent theorems • apply properties of tangents to circles in problems	200–209
14. Trigonometry	Students should be able to: • derive sine rule and apply to solving triangles • derive cosine rule and apply to solving triangles	210–223
15. Bearings	Students should be able to: • revise angles of elevation and depression • use trigonometric ratios and sin and cos rules to problems involving angles of elevation and depression • define and draw cardinal points (4, 8 and 16 points) • use cardinal notation and three-digit notation for bearings • solve practical problems involving bearings	224–235
16. Statistics	Students should be able to: • calculate mean, median, mode and range of ungrouped data • summarise ungrouped data in frequency tables • calculate mean deviation, variance and standard deviation of ungrouped data • determine class intervals, class boundaries and class midpoints of grouped data • summarise grouped data in frequency tables • estimate mean, median and range, mean deviation variance and standard deviation of grouped data • identify modal class of grouped data • represent data in histograms • summarise grouped data in cumulative frequency tables • draw ogives of cumulative frequencies • use ogives to estimate quartiles and percentiles	236–266
17. Probability	Students should be able to: • define and use probability terms • list chance instruments used • list outcomes and solve problems in equiprobable sample space • investigate experimental probability as number of trials increases • apply probability rule for complementary events • apply addition rule for mutually exclusive events • apply multiplication rule for independent events • perform experiments with or without replacement • solve various practical problems involving probability	267–286
18. Revision and exam practice	Students will carry out revision and practice problems on: • number and numeration • algebraic processes • geometry • statistics	287–307

TOPIC 1 Revision

Introduction

This first topic is a revision of the main concepts covered in Mathematics in SS1. We have not attempted to cover all the work that was done in SS1 but instead have focused on the concepts that will be revisited in SS2. The material is arranged by topic and follows a logical progression through the curriculum.

Preparation

Prepare charts that show the main formulae used in this topic and display these around your classroom for your students to refer to.

Before presenting this topic to your class, give careful thought to how you want to work though this revision material. All of the exercises can and should be used at your own discretion. In some cases, you might choose to set your students selected questions only or even to skip an exercise altogether. In other cases, you might prefer to postpone some sections of this revision to a later stage in the year, if you feel that your students will benefit more by doing the revision just before the relevant section of the work is covered in SS2.

Introduction for students

Explain to your class how you have decided to work through this revision material. Some sections will benefit from a class discussion, some sections can be given as homework and some sections can be given as group or pair work. By varying your methodology, you will hold the interest of your students.

Answers

Exercise 1.1

1. **a)** 6 237 714 ≈ 6 240 000 **b)** 552 459 ≈ 550 000
c) 846 010 ≈ 850 000

2. **a)** 46 762 ≈ 46 800 **b)** 93 417 ≈ 93 400
c) 50 094 ≈ 50 100

3. **a)** 17.7108 ≈ 17.7 **b)** 0.9575 ≈ 1.0
c) 31.013 ≈ 31.0

4. **a)** 0.28456 ≈ 0.285 **b)** 17.0407 ≈ 17.041
c) 1.2644 ≈ 1.264

5. **a)** 52 913 900 ≈ 53 000 000 **b)** 1.9523 ≈ 2.0
c) 0.00284 ≈ 0.0028

6. **a)** 41 632 509 ≈ 41 630 000 **b)** 71.20437 ≈ 71.20
c) 0.18947 ≈ 0.1895

Exercise 1.2

1. **a)** $1.86 \times 10^3 = 1\,860$
b) $3.047 \times 10^{-2} = 0.03047$
c) $2.1982 \times 10^7 = 21\,982\,000$
d) $5.58 \times 10^{-5} = 0.0000558$
e) $1.296 \times 10^6 = 1\,296\,000$
f) $8.404 \times 10^{-6} = 0.000008404$
g) $6.0339 \times 10^5 = 603\,390$
h) $9.515 \times 10^{-4} = 0.0009515$

2. **a)** $218.5 = 2.185 \times 10^2$
b) $0.523 = 5.23 \times 10^{-1}$
c) $417\,000 = 4.17 \times 10^5$
d) $0.008272 = 8.272 \times 10^{-3}$
e) $84\,040 = 8.404 \times 10^4$
f) $0.000093658 = 9.3658 \times 10^{-5}$
g) $6\,217\,400 = 6.2174 \times 10^6$
h) $0.00000030011 = 3.0011 \times 10^{-7}$

Exercise 1.3

1. $3 \times p \times 2 \times q \times p \times q \times p = 6p^3q^2$

2. **a)** $x^4 \times x$
$= x^{4+1}$
$= x^5$

b) $z^8 \div z^4$
$= z^{8-4}$
$= z^4$

c) $(xy^2z^3)^0$
$= 1$

d) $a^4 \times a^{-1}$
$= a^{4-1}$
$= a^3$

e) 1^{2+6}
$= 1^8$
$= 1$

f) $(p^2q^{-3})^3$
$= p^{2 \times 3}q^{-3 \times 3}$
$= p^6q^{-9}$
$= \frac{p^6}{q^9}$

g) $(r^2 \times r^{-8})^{-1}$
$= (r^{2-8})^{-1}$
$= (r^{-6})^{-1}$
$= r^{-6 \times -1}$
$= r^6$

h) $\left(\frac{x^2}{y^3}\right)^p$
$= \frac{x^{2 \times p}}{y^{3 \times p}}$
$= \frac{x^{2p}}{y^{3p}}$

i) $x^{-2} \times x^{-8}$
$= x^{-2-8}$
$= x^{-10}$
$= \frac{1}{x^{10}}$

j) $x^2 \div x^{11}$
$= x^{2-11}$
$= x^{-9}$
$= \frac{1}{x^9}$

k) $a^4 \times a^{-4}$
$= a^{4-4}$
$= a^0$
$= 1$

l) $(st)^{-6}$
$= s^{-6}t^{-6}$
$= \frac{1}{s^6t^6}$

m) $(-3pqr)^{-3}$
$= \frac{1}{(-3pqr)^3}$
$= \frac{1}{(-3)^3p^3q^3r^3}$
$= -\frac{1}{27p^3q^3r^3}$

n) $\left(\frac{c^{-2}}{d^3}\right)^{-c}$
$= \frac{c^{-2 \times -c}}{d^{3 \times -c}}$
$= \frac{c^{2c}}{d^{-3c}}$
$= c^{2c}d^{3c}$

o) $81^{\frac{1}{4}}$
$= (3^4)^{\frac{1}{4}}$
$= 3^{4 \times \frac{1}{4}}$
$= 3^1$
$= 3$

p) $(125y^6)^{\frac{2}{3}}$
$= (5^3y^6)^{\frac{2}{3}}$
$= 5^{3 \times \frac{2}{3}}y^{6 \times \frac{2}{3}}$
$= 5^2y^4$

3. **a)** $\sqrt[3]{p}$
$= p^{\frac{1}{3}}$

b) $\sqrt[4]{y^3}$
$= y^{\frac{3}{4}}$

c) $\sqrt{x^7}$
$= x^{\frac{7}{2}}$

d) $\sqrt[a]{4^b}$
$= 4^{\frac{b}{a}}$

4. **a)** $x^{\frac{1}{3}}$
$= \sqrt[3]{x}$

b) $(6b)^{\frac{1}{2}}$
$= \sqrt{6b}$

c) $y^{\frac{2}{n}}$
$= \sqrt[n]{y^2}$

d) $z^{\frac{6}{5}}$
$= \sqrt[5]{z^6}$

Exercise 1.4

1. **a)** $2^5 = 32$
$\therefore \log_2 32 = 5$

b) $3^7 = 2\ 187$
$\therefore \log_3 2\ 187 = 7$

c) $4^p = q$
$\therefore \log_4 q = p$

d) $x^y = z$
$\therefore \log_x z = y$

2. **a)** $\log 10 = 1$
$\therefore 10^1 = 10$

b) $\log_6 216 = 3$
$\therefore 6^3 = 216$

c) $\log_j 5 = k$
$\therefore j^k = 5$

d) $\log_c d = e$
$\therefore c^e = d$

3. **a)** $\log 2 + \log 5$
$= \log (2 \times 5)$
$= \log 10$
$= \log_{10} 10$
$= 1$

b) $\log_3 3 + \log_3 9$
$= \log_3 (3 \times 9)$
$= \log_3 27$
$= \log_3 3^3$
$= 3 \log_3 3$
$= 3(1)$
$= 3$

c) $\log_8 4 + \log_8 16$
$= \log_8 (4 \times 16)$
$= \log_8 64$
$= \log_8 8^2$
$= 2 \log_8 8$
$= 2(1)$
$= 2$

d) $\log_{12} 288 - \log_{12} 2$
$= \log_{12} (288 \div 2)$
$= \log_{12} 144$
$= \log_{12} 12^2$
$= 2 \log_{12} 12$
$= 2(1)$
$= 2$

e) $\log 500 - \log 5 - \log 10^3$
$= \log (500 \div 5) - 3 \log 10$
$= \log_{10} 100 - 3 \log_{10} 10$
$= \log_{10} 10^2 - 3 \log_{10} 10$
$= 2 \log_{10} 10 - 3 \log_{10} 10$
$= 2(1) - 3(1)$
$= -1$

f) $\log_3 \frac{1}{9}$
$= \log_3 \frac{1}{3^2}$
$= \log_3 3^{-2}$
$= -2 \log_3 3$
$= -2(1)$
$= -2$

g) $\log_5 625$
$= \log_5 5^4$
$= 4 \log_5 5$
$= 4(1)$
$= 4$

h) $\log_m m^3 + \log_k k^4 - \log_p p$
$= 3 \log_m m + 4 \log_k k - \log_p p$
$= 3(1) + 4(1) - 1$
$= 6$

i) $\log 16 + 2\log 5 - 2\log 2$
$= \log 16 + \log 5^2 - \log 2^2$
$= \log 16 + \log 25 - \log 4$
$= \log (16 \times 25 \div 4)$
$= \log 100$
$= \log_{10} 100$
$= \log_{10} 10^2$
$= 2\log_{10} 10$
$= 2(1)$
$= 2$

j) $\log_5 27 \times \log_2 25 \times \log_3 16$
$= \left(\frac{\log 27}{\log 5}\right)\left(\frac{\log 25}{\log 2}\right)\left(\frac{\log 16}{\log 3}\right)$
$= \frac{\log 27 \times \log 25 \times \log 16}{\log 5 \times \log 2 \times \log 3}$
$= \frac{\log 3^3 \times \log 5^2 \times \log 2^4}{\log 5 \times \log 2 \times \log 3}$
$= \frac{3\log 3 \times 2\log 5 \times 4\log 2}{\log 5 \times \log 2 \times \log 3}$
$= \frac{24(\log 3 \times \log 5 \times \log 2)}{\log 5 \times \log 2 \times \log 3}$
$= 24$

Exercise 1.5

1. a) $5.945 = 10^{0.7738 + 0.0004}$
$= 10^{0.7742}$
$\therefore \log 5.945 = 0.7742$

b) $59.45 = 5.945 \times 10^1$
$\therefore \log 59.45 = \log 5.945 + 1$
$= 0.7742 + 1$
$= 1.7742$

c) $1.189 = 10^{0.0719 + 0.0034}$
$= 10^{0.0753}$
$\therefore \log 1.189 = 0.0753$

d) $118.9 = 1.189 \times 10^2$
$\therefore \log 118.9 = \log 1.189 + 2$
$= 0.0753 + 2$
$= 2.0753$

e) $4.374 = 10^{0.6405 + 0.0004}$
$= 10^{0.6409}$
$\therefore \log 4.374 = 0.6409$

f) $4\ 374 = 4.374 \times 10^3$

$\therefore \log 4\ 374 = \log 4.374 + 3$

$= 0.6409 + 3$

$= 3.6409$

g) $7.491 = 10^{0.8745 + 0.0001}$

$= 10^{0.8746}$

$\therefore \log 7.491 = 0.8746$

h) $74\ 910 = 7.491 \times 10^4$

$\therefore \log 74\ 910 = \log 7.491 + 4$

$= 0.8746 + 4$

$= 4.8746$

2. **a)** $0.973 = 0.9727 + 0.0003$

A logarithm of 0.973 corresponds to the number 9.397.

$\therefore \log^{-1} 0.973 = 9.397$

b) $\log^{-1} 1.973 = 9.397 \times 10^1$

$= 93.97$

c) $0.1658 = 0.1644 + 0.0014$

A logarithm of 0.1658 corresponds to the number 1.465.

$\therefore \log^{-1} 0.1658 = 1.465$

d) $\log^{-1} 3.1658 = 1.465 \times 10^3$

$= 1\ 465$

e) $0.6612 = 0.6609 + 0.0003$

A logarithm of 0.6612 corresponds to the number 4.583.

$\therefore \log^{-1} 0.6612 = 4.583$

f) $\log^{-1} 2.6612 = 4.583 \times 10^2$

$= 458.3$

g) $0.4202 = 0.4200 + 0.0002$

A logarithm of 0.4202 corresponds to the number 2.631.

$\therefore \log^{-1} 0.4202 = 2.631$

h) $\log^{-1} 4.4202 = 2.631 \times 10^4$

$= 26\ 310$

3. **a)** $\log(6.651 \times 5.12) = \log 6.651 + \log 5.12$
$= 0.8229 + 0.7093$
$= 1.5322$
$\therefore 6.651 \times 5.12 = \log^{-1} 1.5322$
$= 3.405 \times 10^1$
$= 34.05$

b) $\log(99.44 \div 32.31) = \log 99.44 - \log 32.31$
$= 1.9976 - 1.5093$
$= 0.4883$
$\therefore 99.44 \div 32.31 = \log^{-1} 0.4883$
$= 3.078$

c) $\log(33.39 \times 12.95) = \log 33.39 + \log 12.95$
$= 1.5236 + 1.1123$
$= 2.6359$
$\therefore 33.39 \times 12.95 = \log^{-1} 2.6359$
$= 4.324 \times 10^2$
$= 432.4$

d) $\log(839.2 \div 101.2) = \log 839.2 - \log 101.2$
$= 2.9239 - 2.0052$
$= 0.9187$
$\therefore 839.2 \div 101.2 = \log^{-1} 0.9187$
$= 8.293$

e) $\log(74.6 \times 29.69) = \log 74.6 + \log 29.69$
$= 1.8727 + 1.4726$
$= 3.3453$
$\therefore 74.6 \times 29.69 = \log^{-1} 3.3453$
$= 2.215 \times 10^3$
$= 2\,215$

f) $\log(62.3 \div 3.482) = \log 62.3 - \log 3.482$
$= 1.7945 - 0.5419$
$= 1.2526$
$\therefore 62.3 \div 3.482 = \log^{-1} 1.2526$
$= 1.789 \times 10^1$
$= 17.89$

g) $\log(630.7 \times 2.226) = \log 630.7 + \log 2.226$
$= 2.7998 + 0.3476$
$= 3.1474$
$\therefore 630.7 \times 2.226 = \log^{-1} 3.1474$
$= 1.404 \times 10^3$
$= 1\,404$

h) $\log(47.93 \div 8.724) = \log 47.93 - \log 8.724$
$= 1.6806 - 0.9407$
$= 0.7399$
$\therefore 47.93 \div 8.724 = \log^{-1} 0.7399$
$= 5.494$

Exercise 1.6

1. $A = P(1 - \frac{r}{100})^n$
$\therefore P = \frac{A}{(1 - \frac{r}{100})^n}$

2. $SI = \frac{Prt}{100}$
$\therefore \frac{Prt}{100} = SI$
$\therefore Prt = 100SI$
$\therefore t = \frac{100SI}{Pr}$

3. $V = \frac{1}{2}bhH$
$\therefore \frac{1}{2}bhH = V$
$\therefore bhH = 2V$
$\therefore h = \frac{2V}{bH}$

4. $x^2 + y^2 = r^2$
$\therefore r^2 = x^2 + y^2$
$\therefore r = \pm\sqrt{x^2 + y^2}$
$\therefore r = \sqrt{x^2 + y^2}$
(Discard the ± sign in the final answer, as a radius is positive.)

5. $V = s^3$
$\therefore s = \sqrt[3]{V}$

6. $T_n = ar^{n-1}$
$\therefore ar^{n-1} = T_n$
$\therefore a = \frac{T_n}{r^{n-1}}$

7. $S_\infty = \frac{a}{1-r}$
$\therefore S_\infty(1 - r) = a$
$\therefore 1 - r = \frac{a}{S_\infty}$
$\therefore -r = \frac{a}{S_\infty} - 1$
$\therefore r = 1 - \frac{a}{S_\infty}$

8. $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$
$\therefore \frac{1}{v} = \frac{1}{f} - \frac{1}{u}$
$\therefore \frac{1}{v} = \frac{u-f}{fu}$
$\therefore v = \frac{fu}{u-f}$

Exercise 1.7

1. $5x + 4 = 3x + 8$
$\therefore 5x - 3x = 8 - 4$
$\therefore 2x = 4$
$\therefore x = 2$

2. $\frac{3x}{2} - 1 = 5$
$\therefore 3x - 2 = 10$
$\therefore 3x = 10 + 2$
$\therefore 3x = 12$
$\therefore x = 4$

3. $2(x + 6) - 1 = 3 - (7 - x)$
$\therefore 2x + 12 - 1 = 3 - 7 + x$
$\therefore 2x - x = 3 - 7 - 12 + 1$
$\therefore x = -15$

4. $10(2 - x) + 6x = 3(x - 5)$

$\therefore 20 - 10x + 6x = 3x - 15$

$\therefore -10x + 6x - 3x = -15 - 20$

$\therefore -7x = -35$

$\therefore x = 5$

5. $5(2x - 1) = 2(5x + 9)$

$\therefore 10x - 5 = 10x + 18$

$\therefore 10x - 10x = 18 + 5$

$\therefore 0 = 23$

$\therefore$ no solution

6. $\frac{x - 9}{2} = \frac{3x + 4}{4 + 1}$

Multiply throughout by 4:

$\therefore 2(x - 9) = 3x + 4 + 4$

$\therefore 2x - 18 = 3x + 8$

$\therefore 2x - 3x = 8 + 18$

$\therefore -x = 26$

$\therefore x = -26$

7. $2 + \frac{x + 2}{2} = \frac{7 + x}{2} - 0.5$

Multiply throughout by 2:

$\therefore 4 + x + 2 = 7 + x - 1$

$\therefore x - x = 7 - 1 - 4 - 2$

$\therefore 0 = 0$

$\therefore$ identity (true for all values of x)

8. $\frac{3}{x} + 7 - \frac{5}{2x} = \frac{2x - 1}{x}$

Multiply throughout by $2x$:

$\therefore 6 + 14x - 5 = 4x - 2$

$\therefore 14x - 4x = -2 - 6 + 5$

$\therefore 10x = -3$

$\therefore x = -\frac{3}{10}$

9. $\frac{x + 6}{x - 3} = \frac{x - 8}{x + 1}$

Cross-multiply:

$\therefore (x + 6)(x + 1) = (x - 8)(x - 3)$

$\therefore x^2 + x + 6x + 6 = x^2 - 3x - 8x + 24$

$\therefore x + 6x + 3x + 8x = 24 - 6$

$\therefore 18x = 18$

$\therefore x = 1$

10. $\frac{2x+3}{x-4} = \frac{2x-1}{x+3}$

Cross-multiply:

$$\therefore (2x+3)(x+3) = (2x-1)(x-4)$$
$$\therefore 2x^2 + 6x + 3x + 9 = 2x^2 - 8x - x + 4$$
$$\therefore 6x + 3x + 8x + x = 4 - 9$$
$$\therefore 18x = -5$$
$$\therefore x = -\tfrac{5}{18}$$

Exercise 1.8

1. **a)** $x^2 - 5x + 4 = 0$

$\therefore (x-4)(x-1) = 0$

$\therefore x - 4 = 0$ or $x - 1 = 0$

$\therefore x = 4$ or $x = 1$

b) $2x^2 - 24x + 54 = 0$

$\therefore x^2 - 12x + 27 = 0$

$\therefore (x-9)(x-3) = 0$

$\therefore x - 9 = 0$ or $x - 3 = 0$

$\therefore x = 9$ or $x = 3$

c) $6x^2 + 13x + 6 = 0$

$\therefore (3x+2)(2x+3) = 0$

$\therefore 3x + 2 = 0$ or $2x + 3 = 0$

$\therefore x = -\tfrac{2}{3}$ or $x = -\tfrac{3}{2}$

d) $3x^2 - 23x - 8 = 0$

$\therefore (3x+1)(x-8) = 0$

$\therefore 3x + 1 = 0$ or $x - 8 = 0$

$\therefore x = -\tfrac{1}{3}$ or $x = 8$

2. **a)** $x^2 + x - 10 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$, $b = 1$ and $c = -10$

$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10)}}{2(1)}$

$= \frac{-1 \pm \sqrt{1 + 40}}{2}$

$= \frac{-1 \pm \sqrt{41}}{2}$

$\therefore x = 2.70$ or $x = -3.70$

b) $x^2 + 7x + 2 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$, $b = 7$ and $c = 2$

$\therefore x = \frac{-7 \pm \sqrt{7^2 - 4(1)(2)}}{2(1)}$

$= \frac{-7 \pm \sqrt{49 - 8}}{2}$

$= \frac{-7 \pm \sqrt{41}}{2}$

$\therefore x = -0.30$ or $x = -6.70$

c) $2x^2 + 11x + 6 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 2$, $b = 11$ and $c = 6$

$\therefore x = \frac{-11 \pm \sqrt{11^2 - 4(2)(6)}}{2(2)}$

$= \frac{-11 \pm \sqrt{121 - 48}}{4}$

$= \frac{-11 \pm \sqrt{73}}{4}$

$\therefore x = -0.61$ or $x = -4.89$

d) $3x^2 - x - 9 = 0$

$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 3$, $b = -1$ and $c = -9$

$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-9)}}{2(3)}$

$= \frac{1 \pm \sqrt{1 + 108}}{6}$

$= \frac{1 \pm \sqrt{109}}{6}$

$\therefore x = 1.91$ or $x = -1.57$

3. a) $x^2 - 2x - 5 = 0$

$\therefore x^2 - 2x = 5$

$\therefore x^2 - 2x + 1^2 = 5 + 1$

$\therefore (x - 1)^2 = 6$

$\therefore x - 1 = \pm\sqrt{6}$

$\therefore x = 1 \pm \sqrt{6}$

$\therefore x = 3.45$ or $x = -1.45$

b) $x^2 + 8x + 3 = 0$

$\therefore x^2 + 8x = -3$

$\therefore x^2 + 8x + 4^2 = -3 + 16$

$\therefore (x + 4)^2 = 13$

$\therefore x + 4 = \pm\sqrt{13}$

$\therefore x = -4 \pm \sqrt{13}$

$\therefore x = -0.39$ or $x = -7.61$

c) $2x^2 - 9x - 4 = 0$

$\therefore 2x^2 - 9x = 4$

$\therefore x^2 - \frac{9}{2}x = 2$

$\therefore x^2 - \frac{9}{2}x + \left(\frac{9}{4}\right)^2 = 2 + \frac{81}{16}$

$\therefore \left(x - \frac{9}{4}\right)^2 = \frac{32}{16} + \frac{81}{16}$

$\therefore \left(x - \frac{9}{4}\right)^2 = \frac{113}{16}$

$\therefore x - \frac{9}{4} = \pm\frac{\sqrt{113}}{16}$

$\therefore x = \frac{9}{4} \pm \sqrt{\frac{113}{4}}$

$\therefore x = 4.91$ or $x = -0.41$

d) $6x^2 - 18x + 1 = 0$

$\therefore 6x^2 - 18x = -1$

$\therefore x^2 - 3x = -\frac{1}{6}$

$\therefore x^2 - 3x + \left(\frac{3}{2}\right)^2 = -\frac{1}{6} + \frac{9}{4}$

$\therefore \left(x - \frac{3}{2}\right)^2 = -\frac{2}{12} + \frac{27}{12}$

$\therefore \left(x - \frac{3}{2}\right)^2 = \frac{25}{12}$

$\therefore x - \frac{3}{2} = \pm\sqrt{\frac{25}{12}}$

$\therefore x = \frac{3}{2} \pm \sqrt{\frac{25}{12}}$

$\therefore x = 2.94$ or $x = 0.06$

Exercise 1.9

1. a) $y = a(x - x_1)(x - x_2)$

$\therefore y = a(x + 3)(x - 5)$

Substitute $(4; -7)$:

$\therefore -7 = a(4 + 3)(4 - 5)$

$\therefore -7 = a(7)(-1)$

$\therefore -7 = -7a$

$\therefore a = 1$

$\therefore y = (x + 3)(x - 5)$

$\therefore y = x^2 - 5x + 3x - 15$

$\therefore y = x^2 - 2x - 15$

b) $y = a(x - x_1)(x - x_2)$

$\therefore y = a(x + 3)(x - 2)$

Substitute $(-2; 2)$:

$\therefore 2 = a(-2 + 3)(-2 - 2)$

$\therefore 2 = a(1)(-4)$

$\therefore 2 = -4a$

$\therefore a = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}(x + 3)(x - 2)$

$= -\frac{1}{2}(x^2 - 2x + 3x - 6)$

$= -\frac{1}{2}(x^2 + x - 6)$

$\therefore y = -\frac{1}{2}x^2 - \frac{1}{2}x + 3$

2. **a)** $y = -x^2 + 3x + 4$

$a = -1$, so the arms of the quadratic graph go down.

The y intercept is 4.

x intercept(s), set $y = 0$:

$-x^2 + 3x + 4 = 0$

$\therefore x^2 - 3x - 4 = 0$

$\therefore (x - 4)(x + 1) = 0$

$\therefore x - 4 = 0$ or $x + 1 = 0$

$\therefore x = 4$ or $x = -1$

The equation of the line of symmetry:

$x = -\frac{b}{2a}$

$= -\frac{3}{2(-1)}$

$= \frac{3}{2}$

$= 1\frac{1}{2}$

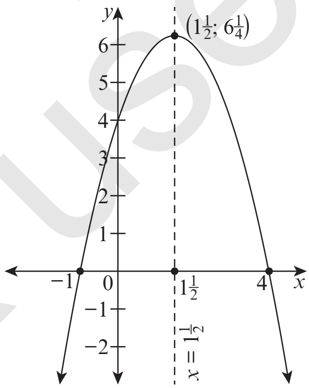
Maximum value:

$y = -x^2 + 3x + 4$

$= -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 4$

$= -\frac{9}{4} + \frac{9}{2} + 4$

$= 6\frac{1}{4}$



b) $y = 3x^2 - 12x + 9$

$a = 3$, so the arms of the quadratic graph go up.

The y intercept is 9.

x intercept(s), set $y = 0$:

$3x^2 - 12x + 9 = 0$

$\therefore x^2 - 4x + 3 = 0$

$\therefore (x - 3)(x - 1) = 0$

$\therefore x - 3 = 0$ or $x - 1 = 0$

$\therefore x = 3$ or $x = 1$

The equation of the line of symmetry:

$x = -\frac{b}{2a}$

$= -\frac{-12}{2(3)}$

$= 2$

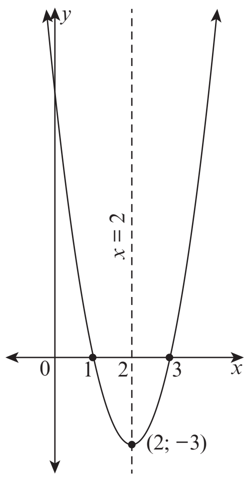
Minimum value:

$y = 3x^2 - 12x + 9$

$= 3(2)^2 - 12(2) + 9$

$= 12 - 24 + 9$

$= -3$



Exercise 1.10

1. **a)**

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

b)

P	Q	P ⇔ Q
T	T	T
T	F	F
F	T	F
F	F	T

2. **a)** It is not Saturday.
b) The market is not busy.
c) It is Saturday and the market is busy.
d) It is Saturday or the market is busy.
e) The market is busy.
f) If it is Saturday, then the market is busy.
g) The market is busy if and only if it is Saturday.
h) If it is Saturday, then the market is not busy.

Exercise 1.11

1. **a)** $C = 2\pi r \times \frac{\theta}{360}$
$= 2 \times \pi \times 20 \times \frac{60}{360}$
$= 20.94$ cm

b) $A = \pi r^2 \times \frac{\theta}{360}$
$= \pi \times 20^2 \times \frac{205}{360}$
$= 715.58 \text{ cm}^2$

c) $C = r + r + 2\pi r \times \frac{\theta}{360}$
$= 20 + 20 + 2 \times \pi \times 20 \times \frac{130}{360}$
$= 85.38$ cm

2. $2\pi r \times \frac{\theta}{360} = r$

$\therefore \frac{\theta}{360} = \frac{r}{2\pi r} = \frac{1}{2\pi}$

$\therefore \theta = \frac{360}{2\pi} = 57.3°$

Exercise 1.12

1. **a)** $\sin 29° = 0.48$ **b)** $\cos 78.3° = 0.20$
c) $\tan 42.7° = 0.92$ **d)** $\sin 65.6° = 0.91$
e) $\cos 38° = 0.79$ **f)** $\tan 85.8° = 13.62$

2. **a)** $\sin\theta = 0.24 \therefore \theta = 13.9°$ **b)** $\cos\theta = 0.43 \therefore \theta = 64.5°$
c) $\tan\theta = 6 \therefore \theta = 80.5°$ **d)** $\sin\theta = 0.75 \therefore \theta = 48.6°$
e) $\cos\theta = 0.57 \therefore \theta = 55.2°$ **f)** $\tan\theta = 1.2 \therefore \theta = 50.2°$

3. **a)** $\tan W\hat{B}T = \frac{WT}{BT}$

$\therefore BT = \frac{15}{\tan 36°}$

$= 20.65$ m

$\therefore AT = 2 \times 20.65$ m

$= 41$ m

b) $\tan A = \frac{WT}{AT}$

$= \frac{15}{41}$

$\therefore \hat{A} = \tan^{-1} \frac{15}{41}$

$= 20°$

4. **a)** $P\hat{Q}N = 180° - 145°$ (coint. ∠s; North lines are parallel)

$= 35°$

$P\hat{Q}R = 60°$ (△PQR is equilateral)

$\therefore$ reflex $N\hat{Q}R = 360° - 35° - 60°$

$= 265°$

∴ The bearing of *r* from Q is 265°.

b) $N\hat{R}Q = 180° - 35° - 60°$ (coint. ∠s; North lines are parallel)

$= 85°$

$\therefore N\hat{R}P = 85° - 60°$

$= 25°$

∴ The bearing of P from *r* is 025°.

Exercise 1.13

1.

$a = 37°$ (alt. ∠s)

$b = 37°$ (isosceles △)

$c = 75°$ (vert. opp. ∠s)

$d = 180° - 37° - 75°$ (sum of ∠s of a △)

$= 68°$

$e = 180° - 37° - 37° - 68°$ (sum of ∠s of a △)

$= 38°$

$f = 38°$ (alt. ∠s)

$g = 180° - 75°$ (∠s on a straight line)

$= 105°$

$h = 105°$ (vert. opp. ∠s)

$i = 180° - 68° - 38°$ (∠s on a straight line)

$= 74°$

2. **a)** △ABC ||| △BDC ||| △ADB

b) AB : BD : AD = 7 : 6 : 3

∴ AC : BC : AB = 7 : 6 : 3

∴ BC : CD : BD = 7 : 6 : 3

(i) $\frac{AC}{BC} = \frac{7}{6}$

$\therefore \frac{AC}{51} = \frac{7}{6}$

$\therefore AC = \frac{51 \times 7}{6}$

$= 59.5$ mm

(ii) $\frac{AB}{BC} = \frac{3}{6} = \frac{1}{2}$

$\therefore \frac{AB}{51} = \frac{1}{2}$

$\therefore AB = \frac{51}{2}$

$= 25.5$ mm

(iii) $\frac{CD}{BC} = \frac{6}{7}$

$\therefore \frac{CD}{51} = \frac{6}{7}$

$\therefore CD = \frac{51 \times 6}{7}$

$= 43.7$ mm

(iv) $\frac{BD}{AB} = \frac{6}{7}$

$\therefore \frac{BD}{25.5} = \frac{6}{7}$

$\therefore BD = \frac{25.5 \times 6}{7}$

$= 21.9$ mm

3. **a)** $\frac{RU}{UV} = \frac{RT}{ST} = \frac{1}{2}$ (In △RSV, TU ‖ SV.)

b) $\frac{RU}{RV} = \frac{1}{1 + 2} = \frac{1}{3}$

c) $\frac{RU}{PR} = \frac{1}{3 + 3} = \frac{1}{6}$ (PV = RV, diag. of parm bisect one another)

d) $\frac{UV}{PV} = \frac{2}{3}$

Exercise 1.14

1. **a)** 5 **b)** 卌 |||| **c)** 8 **d)** || **e)** 40

2. 40 applicants applied for the position.

3. Three applicants had no previous experience.

4. **a)** The sum of all the data values is $(0 \times 3) + (1 \times 5) + (2 \times 7) + (3 \times 9) + (4 \times 8) + (5 \times 6) + (6 \times 2) = 120$. There are 40 data values. ∴ The mean is $\frac{120}{40} = 3$ jobs.

b) There are 40 data values in the data set. The middle value lies halfway between the 20th and the 21st data values, which are 3 and 3. ∴ The median is 3 jobs.

c) The mode is 3 jobs because this is the data value with the highest frequency (9).

d) The range = 6 − 0 = 6 jobs.

Exercise 1.15

1. First calculate the midpoints of the classes:

Age (years)	21–25	26–30	31–35	36–40	41–45	46–50
Midpoint	23	28	33	38	43	48
Frequency	12	15	6	4	2	1

The estimated sum of all the data values is $(23 \times 12) + (28 \times 15) + (33 \times 6) + (38 \times 4) + (43 \times 2) + (48 \times 1) = 1\,180$ years. There are 40 data values. ∴ The estimated mean $= \frac{1\,180}{40} = 29.5$ years.

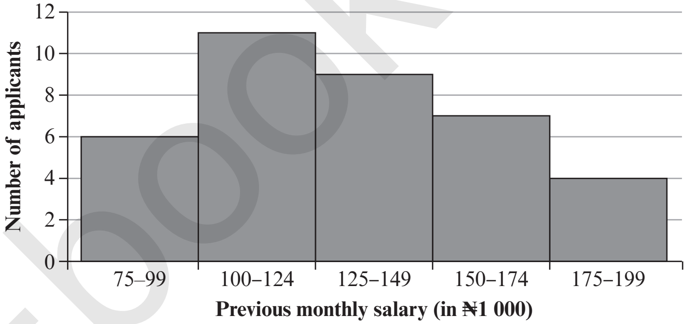
2. There are 40 data values in the data set. The middle value lies halfway between the 20th and the 21st data values. ∴ The median is the 20.5th data value.
The 20.5th data value is the 8.5th data value in the class 26−30 (because 20.5 − 12 = 8.5).
∴ The estimated median = $25.5 + \frac{8.5}{15} \times 5 = 28.3$ years, (correct to one decimal place). (We use the lower class boundary, 25.5 years, to get the most accurate estimate.)

3. The modal class is 26−30 years, because it has the highest frequency.

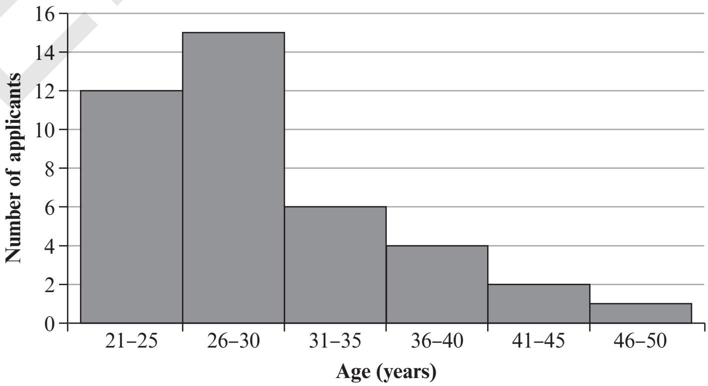
4. The largest possible range = 50 − 21 = 29 years.
The smallest possible range = 46 − 25 = 21 years.
So all we can say for certain is that the range lies between 21 years and 29 years.

Exercise 1.16

1.



2.



TOPIC 2 Logarithms

Introduction

We begin this topic by revising last year's work.

Then we study the laws of logarithms and see how they are similar to the laws of indices, apply the laws of logarithms to simplify expressions, and also learn the change of base law. Go through the steps to perform mathematical operations to evaluate expressions that are given in the Student's Book, and make sure that students grasp the principles set out.

We then move on to using our knowledge of indices and logarithms to solve logarithmic equations. A logarithmic equation is an equation that contains one or more logarithms. We can use the four laws of logarithms to help us to solve logarithmic equations.

Lastly, we check the accuracy of results by comparing the results obtained from reading values off the logarithm tables and calculating the answer using a scientific calculator.

Common difficulties

The same difficulties that students encounter with indices occur in this section of work. Students need to understand the definition of a logarithm very clearly. They also need to be able to change powers from index form to logarithm form or logarithm form to index form.

Students need to practise writing numbers in standard form as they often write the incorrect power of 10.

Remind students that the base of a logarithm has to be a positive real number but may not equal 1. You may also only take the logarithm of a positive number and the logarithm of 1 is always equal to zero. Make sure the students understand why this is the case.

- Any logarithm $\log_a a = 1$ as $a^1 = a$ for any value of $a > 0$, $a \neq 1$.
- The logarithm of 1 with any positive base always equals 0. $\log_m 1 = 0$ (where $m > 0$ and $m \neq 1$).

Students do not always read the logarithm tables properly. When they are finding the log of a number, use the logarithm tables. When they convert back to the answer, use the antilogarithm tables. If they have access to scientific calculators, they should use these to check their answers.

Preparation

The following charts could be displayed in the classroom as you work through this topic:

- index rules and the logarithm definition and laws
- logarithm and antilogarithm tables
- some powers of ten
- steps for performing mathematical operations to evaluate expressions
- solution chart of logarithms.

You could also create charts from the following information.

Introduction for students

Revise the definition of a logarithm and show the students that the logarithm is the index to which you raise the base. Use some numerical examples:

$8 = 2^3$ which means that $\log_2 8 = 3$

$\log_4 64 = 3$ which means that $4^3 = 64$

Ask the students to think of their own examples and practise changing index form to log form and log form to index form.

Revise calculations with logarithms of numbers greater than 1 and explain that this year we will be using logs to perform calculations on numbers less than 1 using the log tables.

Answers

Exercise 2.1

1 **a)** $504\ 010 = 5.0401 \times 10^5$
b) $0.061 = 6.1 \times 10^{-2}$
c) $0.101 = 1.01 \times 10^{-1}$
d) $0.0014253 = 1.4253 \times 10^{-3}$
e) $0.0000987 = 9.87 \times 10^{-5}$
f) $100\ 000\ 000 = 1 \times 10^8$

g) $0.00020304 = 2.0304 \times 10^{-4}$
h) $12\,856.023 = 1.2856023 \times 10^{4}$
i) $0.000000181 = 1.81 \times 10^{-7}$

2. 178.5 million $= 178\,500\,000 = 1.785 \times 10^{8}$

3. $1\,112\,000\,000 = 1.112 \times 10^{9}$

4. 16%

Exercise 2.2

1. a) 3 b) 6 c) $\frac{5}{4}$
d) 4 e) 4 f) $2 - 2 = 0$
g) $\frac{3}{2}$ h) $\frac{2+2}{2} = 2$

2. a) $\frac{\log 24}{\log 3}$ b) $\frac{\log 54}{\log 2}$
c) $\log_3 81 = \frac{\log 81}{\log 3}$ d) $\frac{\log 27}{\log 3} \times \frac{\log 81}{\log 3}$
e) $\log_6 36 = \frac{\log 36}{\log 6}$ f) $\log_4 1\,024 = \frac{\log 1\,024}{\log 4}$

3. a) $\log_4 (xy)$ b) $\log x^3$ c) 12
d) $\log_5 y^4$ e) x f) $\frac{9+7}{8} = 2$

Exercise 2.3

1. a) 0 b) 3 c) 4 d) 1

2. a) .2567 b) .1881 c) .9581 d) .8263

3. a) 2.7104 b) 1.8418 c) 0.9509 d) 0.2608

4. a) 1 000 b) 243.2 c) 10.64 d) 2.606

Exercise 2.4

1. 15.28
2. 37.088
3. 2.6335
4. 8.346
5. 1.1776
6. 4.2515
7. 2.086
8. 1.2935
9. 0.2957
10. 0.01366

Exercise 2.5

1. a) $0.65 = 10^{\bar{1}.8129}$ b) $0.347 = 10^{\bar{1}.5403}$
c) $0.801 = 10^{\bar{1}.9036}$ d) $0.0632 = 10^{\bar{2}.8007}$
e) $0.0922 = 10^{\bar{2}.9647}$ f) $0.00879 = 10^{\bar{3}.944}$
g) $0.000701 = 10^{\bar{4}.8457}$ h) $0.000123 = 10^{\bar{4}.0899}$

2. **a)** $10^{\bar{1}.203} = 0.1596$ **b)** $10^{\bar{2}.651} = 0.04477$
c) $10^{\bar{3}.896} = 0.00787$ **d)** $10^{\bar{4}.091} = 0.0001233$
e) $10^{\bar{1}.654} = 0.4508$ **f)** $10^{\bar{3}.513} = 0.003258$
g) $10^{\bar{2}.011} = 0.01026$ **h)** $10^{\bar{5}.753} = 0.00005662$

Exercise 2.6

1 **a)** 1.72 **b)** 0.000644 **c)** 16.81 **d)** 0.00001282

2. **a)** 0.8326 **b)** 0.146 **c)** 1.006 **d)** 3.125

3. **a)** 0.4117 **b)** 0.8565 **c)** 47.98 **d)** 0.2284

4. **a)** 0.1954 **b)** 0.0662 **c)** 0.9529 **d)** 0.474

5. **a)** 0.000008742 **b)** 0.000799
c) 88.34 **d)** 1.258

Exercise 2.7

1 **a)** 625 **b)** 1 000 **c)** 12 **d)** 729 or $\frac{1}{729}$
e) 12 **f)** 17 **g)** 2 **h)** 2
i) 3 **j)** 15 625

2. **a)** 4 **b)** 3 **c)** 4 **d)** 24
e) 10 000 **f)** 14 **g)** 5.807 **h)** 7 776
i) 5 **j)** $\frac{3}{2}$

Exercise 2.8

1. **a)** log 3.014 = 0.4792
b) log 0.04625 = $\bar{2}.6651 = 0.6651 - 2 = -1.3349$
c) $\bar{1}.8657 + 0.2065 = 0.0722$
antilog 0.0722 = 1.181
d) $1.6396 - 0.9188 = 0.7208$
antilog 0.7208 = 5.258
e) $3 \times 0.6043 = 1.8129$
antilog 1.8129 = 65.00
f) $2 \times 1.1826 \div 5 = 0.47304$
antilog 0.473 = 2.972

2. **a)** log 3.014 = 0.479143248
b) log 0.04625 = −1.334888263
c) $0.734 \times 1.609 = 1.181006$
d) $43.61 \div 8.293 = 5.258651875$
e) $(4.021)^3 = 65.01330126$
f) $\sqrt[5]{(15.23)^2} = 2.972213219$

3. Note: We are only interested in the percentage *difference*, so negative answers have been given as positive.
   a) $(0.4792 - 0.479143248) \div 0.4792 \times 100 = 0.0118\%$
   b) $[-1.3349 - (-1.334888263)] \div (-1.3349) \times 100 = 0.000879\%$
   c) $(1.181 - 1.181006) \div 1.181 \times 100 = 0.000508\%$
   d) $(5.258 - 5.258651875) \div 5.258 \times 100 = 0.000124\%$
   e) $(65.00 - 65.01330126) \div 65 \times 100 = 0.02046\%$
   f) $(2.972 - 2.972213219) \div 2.972 \times 100 = 0.0072\%$

Assess your progress

1.

Mercury	58 million km = 58 000 000 = 5.8×10^7
Venus	108 million km = 108 000 000 = 1.08×10^8
Earth	150 million km = 150 000 000 = 1.5×10^8
Mars	228 million km = 228 000 000 = 2.28×10^8
Jupiter	778 million km = 778 000 000 = 7.78×10^8
Saturn	1 430 million km = 1 430 000 000 = 1.43×10^9
Uranus	2 870 million km = 2 870 000 000 = 2.87×10^9
Neptune	4 498 million km = 4 498 000 000 = 4.498×10^9

2. **a)** $7^2 = 49$ **b)** $\log_p q = 4$

3. **a)** $\frac{1}{2}$ **b)** -2 **c)** $\frac{1}{5\,000}$ **d)** -2

4. **a)** 5 **b)** 2 **c)** 3 **d)** 1
e) 2 **f)** 3 **g)** $\frac{1}{2}$ **h)** 6
i) $\frac{5}{4}$ **j)** 0 **k)** 2 **l)** 1

5. **a)** 0.0475 **b)** 0.277 **c)** 0.537 **d)** 0.236

6. 2.42

7. 1.183

8. **a)** 2.096 **b)** -1 **c)** 2.86 **d)** 2.68

9. **a)** $x = 0$ (reject $x = -3$) **b)** $x = 69$

10. **a)** 0.0407 **b)** 0.040652507
c) $0.0407 - 0.040652507 = 0.000047493$
This gives a percentage difference of 0.1168% between the answers.

TOPIC 3

Approximations

Introduction

We begin this topic by rounding off numbers to find approximate numbers. The next section of work is on estimation which is a useful skill. Students should estimate their answers to make sure that they look right and make sense. Make sure that the students know at least the first 15 square numbers so that they can estimate square roots.

Students need to understand upper and lower bounds to understand the effect of rounding off numbers.

The last section of work is on calculating the percentage error between an actual amount and an estimated amount.

Common difficulties

In this section of work, it is not easy to know what the 'correct' answer is as the answers of different students may vary. The important thing is that the student's answer should be not too far off the actual answer that would have been found using a calculator or tables.

Preparation

Have interesting data available from local health clinics, schools in the area and the economy, such as financial reports, budgets and population figures. These can be obtained from local newspapers or the internet.

Introduction for students

Practise rounding off really big numbers to the nearest million or trillion. Note the difference between rounding numbers to a given number of decimal places and a given number of significant figures.

Ask the students to think of situations where rounding off numbers is appropriate, for example, buying bricks to build a wall or buying tins of paint.

Answers

Exercise 3.1

1. **a)** 64 000 **b)** 123 000 **c)** 7 000

2. **a)** 4 895 000 000 **b)** 9 016 000 000 **c)** 1 575 000 000

3. **a)** 12.73 **b)** 5.98 **c)** 0.06

4. **a)** 2.072 **b)** 0.732 **c)** 0.007

5. **a)** 59 000 **b)** 8.5 **c)** 210

6. **a)** 0.0101 **b)** 0.0654 **c)** 0.00205

7. **a)** $2.5 \times 365 \times 5 = 4\,562.5$ million barrels
b) 5 trillion

Exercise 3.2

1. **a)** 330 **b)** 130 **c)** 2400 **d)** 1.6
e) 100 **f)** $\frac{7}{5}$ or 1.4 **g)** 5 **h)** 30

2. **a)** 340 g **b)** 180 km/h **c)** 500 g **d)** 1.2 billion

3. **a)** 100 cm^2 **b)** 180 mm or 18 cm
c) 2 **d)** 40

4. **a)** Area $= \frac{1}{2} \times 11 \times 4 = 22$ cm^2
b) NR = 8m (Pythagoras). TR $= \sqrt{73} \approx 8.5$ m

5. **a)** 5.5 **b)** 13.2

6. 50 000 yards **7.** 45 cm^3 **8.** 235 cm^2

9. Circumference ≈ 200 cm. Distance = 200 × 895
≈ 180 000 cm = 1.8 km.

10. 196

Exercise 3.3

1. **a)** **(i)** 7.5 and 8.5 **(ii)** $7.5 \leq x < 8.5$
b) **(i)** 6.25 and 6.35 **(ii)** $6.25 \leq x < 6.35$
c) **(i)** 0.85 and 0.95 **(ii)** $0.85 \leq x < 0.95$
d) **(i)** −0.45 and −0.35 **(ii)** $-0.45 \leq x < -0.35$

2. **a)** 63.75 and 63.85 **b)** $63.75 \leq m < 63.85$

3. lower bound = 5.25 m and upper bound = 5.35 m

4. **a)** lower bound = 48.65 seconds and upper bound = 48.75 seconds
   b) $48.65 \leq t < 48.75$

5. lower bound = 365.245 days and upper bound = 365.255 days

Exercise 3.4

1. **a)** 20% **b)** 7.9% **c)** 25%
   d) 5.2% **e)** 4.3% **f)** 20%

2. 0.05%

3. 7.2%

4. 0.032%

5. 2.8%

6. 3%

Assess your progress

1 **a)** 400 **b)** 430 **c)** 90

2. ₦4 180 million

3. ₦4 500

4. **a)** 1.25% **b)** 2.5%

5. **a)** $\pi \times 9^2 - \pi \times 3^2 = 72\pi \approx 216 \text{ mm}^2$
   b) $(20 \times 14) - (16 \times 10) = 280 - 160 = 120 \text{ cm}^2$
   c) $(22 \times 11) - (10 \times 6) = 242 - 60 = 182 \text{ cm}^2$

6. **a)** 4.5 h
   b) speed = 260 ÷ 4.5 = 57.8 km/h = 60 km/h to one s.f.

7. 0.04%

8. lower bound = 3.5 kg and upper bound = 4.5 kg

9. 1 110 million

10. 0.04%

11. ₦2 trillion

12. 30 025 billion

TOPIC 4
Sequences and series

Introduction

The first section of this topic provides a brief overview of number sequences in general. Then we focus on arithmetic and geometric progressions.

Your students will learn how to find the general term and the sum of arithmetic and geometric progressions, as well as how to apply their knowledge of progressions to solving problems in real-life situations.

Common difficulties

Students usually enjoy working with number patterns. However, they may get confused by what formula to use in any given situation. In order to select the correct formula, students need to engage carefully with what is being asked. Once they have selected the correct formula, they should be very clear about what is given and what unknown variable they need to calculate.

Below are two examples of questions that students generally find confusing:

- What is the value of the 100th term?
- What term is equal to 100?

To answer the first question, students must calculate the value of T_{100}. To answer the second question, students must solve for n in the equation $T_n = 100$.

Preparation

Make a large chart that shows all the formulae that your students will use in this topic. Write these formulae under two headings: 'Arithmetic progressions' and 'Geometric progressions'. Include a description of each formula. Display this chart in your classroom for your students to refer to when needed.

Introduction for the students

Write some simple number patterns on the board and ask your class to comment on what they notice about each one. Make sure that you include examples of arithmetic progressions and geometric progressions, as well as examples that are neither arithmetic nor geometric progressions. You could also ask a different volunteer each time to extend each pattern by writing the next three terms on the board.

Answers

Exercise 4.1

1. **a)** 10, 12, 14 **b)** 91, 89, 87 **c)** 324, 972, 2 916
   d) $1, \frac{1}{3}, \frac{1}{9}$ **e)** −6, 7, −8 **f)** 25, 36, 49
   g) 13, 21, 34 **h)** 38, 51, 66
   i) $5a - 16b, 6a - 32b, 7a - 64b$
   j) p^5, p^6, p^7

2. **a)** Add 2 to each term to get the next term.
   b) Subtract 2 from each term to get the next term.
   c) Multiply each term by 3 to get the next term.
   d) Divide each term by 3 to get the next term.
   e) This sequence consists of all the natural numbers, where every even number is negative.
   f) Square each term.
   g) From the third term onwards, each term is the sum of the two preceding terms.
   h) Add 3, then 5, then 7, and so on. So add two more each time to get the text term.
   i) Add 1 to the coefficient of *a* and multiply the coefficient of *b* by 2 to get the next term.
   j) Multiply each term by *p* to get the next term.

Exercise 4.2

1. This is an arithmetic progression. The common difference is 5.

2. This is not an arithmetic progression.

3. This is an arithmetic progression. The common difference is −5.

4. This is an arithmetic progression. The common difference is 3.

5. This is not an arithmetic progression.

6. This is an arithmetic progression. The common difference is −7.

7. This is not an arithmetic progression.

8. This is not an arithmetic progression.

9. This is an arithmetic progression. The common difference is y.

10. This is an arithmetic progression. The common difference is $(p - q)$.

Exercise 4.3

1. **a)** 3, 12, 21, 30, 39 **b)** 71, 61, 51, 41, 31
   c) 8.1, 7.9, 7.7, 7.5, 7.3 **d)** −25, −22, −19, −16, −13
   e) xy, $3xy$, $5xy$, $7xy$, $9xy$

2. **a)** 24, 31, 38 **b)** −11, −15, −19
   c) −28, −10, 8 **d)** 6.3, 6.5, 6.7
   e) q, $-p + 2q$, $-2p + 3q$

Exercise 4.4

1. **a)** $a = 1$ and $d = 7$
   $\therefore \mathrm{T}_n = a + (n - 1)d = 1 + (n - 1)(7) = 1 + 7n - 7$
   $= 7n - 6$
   b) $a = 13$ and $d = -3$
   $\therefore \mathrm{T}_n = a + (n - 1)d = 13 + (n - 1)(-3) = 13 - 3n + 3$
   $= 16 - 3n$
   c) $a = -11$ and $d = 2$
   $\therefore \mathrm{T}_n = a + (n - 1)d = -11 + (n - 1)(2) = -11 + 2n - 2$
   $= 2n - 13$
   d) $a = 2$ and $d = \frac{1}{2}$
   $\therefore \mathrm{T}_n = a + (n - 1)d = 2 + (n - 1)\left(\frac{1}{2}\right) = 2 + \frac{1}{2}n - \frac{1}{2}$
   $= \frac{1}{2}n + 1\frac{1}{2}$
   e) $a = 42$ and $d = -2.5$
   $\therefore \mathrm{T}_n = a + (n - 1)d = 42 + (n - 1)(-2.5)$
   $= 42 - 2.5n + 2.5 = 44.5 - 2.5n$

f) $a = x$ and $d = x$
$\therefore T_n = a + (n-1)d = x + (n-1)(x) = x + nx - x = nx$

2. a) $a = 11$ and $d = 3$
$\therefore T_n = a + (n-1)d = 11 + (n-1)(3) = 11 + 3n - 3$
$= 3n + 8$

b) $a = 1$ and $d = 7$
$\therefore T_n = a + (n-1)d = 1 + (n-1)(7) = 1 + 7n - 7$
$= 7n - 6$

c) $a = -4$ and $d = 5$
$\therefore T_n = a + (n-1)d = -4 + (n-1)(5) = -4 + 5n - 5$
$= 5n - 9$

d) $a = 39$ and $d = -10$
$\therefore T_n = a + (n-1)d = 39 + (n-1)(-10)$
$= 39 - 10n + 10 = 49 - 10n$

e) $a = -6$ and $d = 0.4$
$\therefore T_n = a + (n-1)d = -6 + (n-1)(0.4)$
$= -6 + 0.4n - 0.4 = 0.4n - 6.4$

f) $a = 8$ and $d = -\frac{1}{2}$
$\therefore T_n = a + (n-1)d = 8 + (n-1)\left(-\frac{1}{2}\right) = 8 - \frac{1}{2}n + \frac{1}{2}$
$= 8\frac{1}{2} - \frac{1}{2}n$

Exercise 4.5

1. a) $T_n = 2n + 6$
$\therefore T_1 = 2(1) + 6 = 8$
$d = T_2 - T_1$
$T_2 = 2(2) + 6 = 10$
$\therefore d = 10 - 8 = 2$
$\therefore$ The first term is 8 and the common difference is 2.

b) $T_n = -n + 2.5$
$\therefore T_1 = -1 + 2.5 = 1.5$
$d = T_2 - T_1$
$T_2 = -2 + 2.5 = 0.5$
$\therefore d = 0.5 - 1.5 = -1$
$\therefore$ The first term is 1.5 and the common difference is -1.

c) $T_n = -3n - 10$
$\therefore T_1 = -3(1) - 10 = -13$
$d = T_2 - T_1$
$T_2 = -3(2) - 10 = -16$
$\therefore d = -16 - (-13)$
$= -3$
$\therefore$ The first term is -13 and the common difference is -3.

d) $T_n = \frac{1}{2}n + \frac{3}{4}$
$\therefore T_1 = \frac{1}{2}(1) + \frac{3}{4} = 1\frac{1}{4}$
$d = T_2 - T_1$
$T_2 = \frac{1}{2}(2) + \frac{3}{4} = 1\frac{3}{4}$
$\therefore d = 1\frac{3}{4} - 1\frac{1}{4} = \frac{1}{2}$
$\therefore$ The first term is $1\frac{1}{4}$ and the common difference is $\frac{1}{2}$.

e) $T_n = 15 - 13n$
$\therefore T_1 = 15 - 13(1) = 2$
$d = T_2 - T_1$
$T_2 = 15 - 13(2) = -11$
$\therefore d = -11 - 2 = -13$
$\therefore$ The first term is 2 and the common difference is -13.

f) $T_n = 1.6n - 0.8$
$\therefore T_1 = 1.6(1) - 0.8$
$= 0.8$
$d = T_2 - T_1$
$T_2 = 1.6(2) - 0.8 = 2.4$
$\therefore d = 2.4 - 0.8 = 1.6$
$\therefore$ The first term is 0.8 and the common difference is 1.6.

2. **a)** $5n - 21 = 99$
$\therefore 5n = 120$
$\therefore n = 24$
$\therefore$ There are 24 terms in the progression.

b) $-n + 3 = -27$
$\therefore -n = -30$
$\therefore n = 30$
$\therefore$ There are 30 terms in the progression.

c) $-7n + 9 = -327$
$\therefore 7n = -336$
$\therefore n = 48$
$\therefore$ There are 48 terms in the progression.

d) $3.5n + 8.5 = 166$
$\therefore 3.5n = 157.5$
$\therefore n = 45$
$\therefore$ There are 45 terms in the progression.

3. **a)** $T_n = 2n + 13$
$\therefore T_{61} = 2(61) + 13$
$= 135$

b) $2n + 13 = 61$
$\therefore 2n = 48$
$\therefore n = 24$
$\therefore$ The 24th term is equal to 61.

4. **a)** $T_n = a + (n - 1)d$
$\therefore T_{12} = a + 11d$
$\therefore a + 11d = -99$ ①
$T_{30} = a + 29d$
$\therefore a + 29d = 63$ ②
Subtract (1) from ②:
$\therefore 18d = 162$
$\therefore d = 9$
Substitute 9 for d in ①:
$a + 11(9) = -99$
$\therefore a = -198$
$\therefore$ The first term is -198 and the common difference is 9.

b) $T_n = a + (n - 1)d$

$\therefore T_{17} = -198 + 16(9) = -54$

c) $T_n = a + (n - 1)d = 0$

$\therefore -198 + (n - 1)(9) = 0$

$\therefore -198 + 9n - 9 = 0$

$\therefore 9n = 207$

$\therefore n = 23$

$\therefore$ There are 23 terms in the progression.

5. $d = T_{32} - T_{31}$

$= 2 - 10 = -8$

$T_n = a + (n - 1)d$

$T_{32} = a + 31(-8)$

$\therefore 2 = a - 248$

$\therefore a = 250$

$\therefore T_{16} = 250 + (16 - 1)(-8) = 130$

Exercise 4.6

1. a) Mean $= \frac{1}{2}(10 + 48) = \frac{58}{2} = 29$

b) Mean $= \frac{1}{2}(-14 + 0) = \frac{-14}{2} = -7$

c) Mean $= \frac{1}{2}(-21 + 19) = \frac{-2}{2} = -1$

d) Mean $= \frac{1}{2}\left(\frac{1}{4} + \frac{3}{4}\right) = \frac{1}{2}$

e) Mean $= \frac{1}{2}(0.6 + 3.5) = \frac{4.1}{2} = 2.05$

f) Mean $= \frac{1}{2}(-12.2 + 16.8) = \frac{4.6}{2} = 2.3$

g) Mean $= \frac{1}{2}(-362 - 204) = \frac{-566}{2} = -283$

h) Mean $= \frac{1}{2}\left(\frac{5}{6} + \frac{6}{7}\right) = \frac{1}{2}\left(\frac{71}{42}\right) = \frac{71}{84}$

2. This arithmetic progression is of the form 32, …, …, 53.

$a = 32$ and $T_4 = 53$

$T_n = a + (n - 1)d$

$\therefore 32 + (4 - 1)d = 53$

$\therefore 3d = 21$

$\therefore d = 7$

$T_2 = 32 + 7 = 39$, $T_3 = 39 + 7 = 46$

$\therefore$ The means are 39 and 46.

3. This arithmetic progression is of the form −18.5, …, …, …, …, …, −5.

$a = -18.5$ and $T_7 = -5$

$$T_n = a + (n - 1)d$$
$$\therefore -18.5 + (7 - 1)d = -5$$
$$\therefore 6d = 13.5$$
$$\therefore d = 2.25$$

$T_2 = -18.5 + 2.25 = -16.25$; $T_3 = -16.25 + 2.25 = -14$;
$T_4 = -14 + 2.25 = -11.75$; $T_5 = -11.75 + 2.25 = -9.5$;
$T_6 = -9.5 + 2.25 = -7.25$
$\therefore$ The means are −16.25, −14, −11.75, −9.5 and −7.25.

Exercise 4.7

1. **a)** $a = 2$, $d = 9$ and $n = 20$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{20} = \frac{20}{2}[2(2) + (20 - 1)(9)] = 10(4 + 171) = 1\,750$

b) $a = 40$, $d = -10$ and $n = 20$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{20} = \frac{20}{2}[2(40) + (20 - 1)(-10)] = 10(80 - 190)$
$= -1\,100$

c) $a = 7.5$, $d = 6.5$ and $n = 20$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{20} = \frac{20}{2}[2(7.5) + (20 - 1)(6.5)] = 10(15 + 123.5)$
$= 1\,385$

d) $a = 342$, $d = -40$ and $n = 20$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{20} = \frac{20}{2}[2(342) + (20 - 1)(-40)] = 10(684 - 760)$
$= -760$

e) $a = -51$, $d = 3$ and $n = 20$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{20} = \frac{20}{2}[2(-51) + (20 - 1)(3)] = 10(-102 + 57)$
$= -450$

f) $a = 63$, $d = -7$ and $n = 20$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{20} = \frac{20}{2}[2(63) + (20 - 1)(-7)] = 10(126 - 133)$
$= -70$

2. **a)** $a = 17$, $d = -1$ and $n = 49$
$S_n = \frac{n}{2}[2a + (n - 1)d]$
$\therefore S_{49} = \frac{49}{2}[2(17) + (49 - 1)(-1)] = \frac{49}{2}(34 - 48) = -343$

b) $a = 92$, $d = 18$ and $n = 49$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{49} = \frac{49}{2}[2(92) + (49 - 1)(18)] = \frac{49}{2}(184 + 864)$

$= 25\,676$

c) $a = 64.2$, $d = -0.4$ and $n = 49$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{49} = \frac{49}{2}[2(64.2) + (49 - 1)(-0.4)] = \frac{49}{2}(128.4 - 19.2)$

$= 2\,675.4$

d) $a = -750$, $d = 25$ and $n = 49$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{49} = \frac{49}{2}[2(-750) + (49 - 1)(25)]$

$= \frac{49}{2}(-1\,500 + 1\,200) = -7\,350$

e) $a = 2.5$, $d = -1.5$ and $n = 49$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{49} = \frac{49}{2}[2(2.5) + (49 - 1)(-1.5)] = \frac{49}{2}(5 - 72)$

$= -1\,641.5$

f) $a = -11$, $d = 2.1$ and $n = 49$

$S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{49} = \frac{49}{2}[2(-11) + (49 - 1)(2.1)] = \frac{49}{2}(-22 + 100.8)$

$= 1\,930.6$

3. **a)** $a = 15$, $n = 35$ and $l = 899$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{35} = \frac{35}{2}(15 + 899) = \frac{35}{2}(914) = 15\,995$

b) $a = -22$, $n = 17$ and $l = 122$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{17} = \frac{17}{2}(-22 + 122) = \frac{17}{2}(100) = 850$

c) $a = 4.4$, $n = 27$ and $l = 85$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{27} = \frac{27}{2}(4.4 + 85) = \frac{27}{2}(89.4) = 1\,206.9$

d) $a = -183$, $n = 15$ and $l = -1$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{15} = \frac{15}{2}(-183 - 1) = \frac{15}{2}(-184) = -1\,380$

e) $a = -0.3$, $n = 61$ and $l = 23.7$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{61} = \frac{61}{2}(-0.3 + 23.7) = \frac{61}{2}(23.4) = 713.7$

f) $a = 192$, $n = 35$ and $l = 362$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{35} = \frac{35}{2}(192 + 362) = \frac{35}{2}(554) = 9\ 695$

4. **a)** $a = 1$, $l = 999$ and $n = 999$

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{999} = \frac{999}{2}(1 + 999) = 499.5(1\ 000) = 499\ 500$

b) $a = 3$ and $d = 6$

$T_n = a + (n - 1)d = 195$

$\therefore 3 + (n - 1)(6) = 195$

$\therefore 3 + 6n - 6 = 195$

$\therefore 6n = 198$

$\therefore n = 33$

$\therefore$ There are 33 terms in the progression.

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{33} = \frac{33}{2}(3 + 195) = 16.5(198) = 3\ 267$

c) $a = 1.25$ and $d = 0.25$

$T_n = a + (n - 1)d = 23.5$

$\therefore 1.25 + (n - 1)(0.25) = 23.5$

$\therefore 1.25 + 0.25n - 0.25 = 23.5$

$\therefore 0.25n = 22.5$

$\therefore n = 90$

$\therefore$ There are 90 terms in the progression.

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{90} = \frac{90}{2}(1.25 + 23.5) = 45(24.75) = 1\ 113.75$

d) $a = -15$ and $d = -2$

$T_n = a + (n - 1)d = -73$

$\therefore -15 + (n - 1)(-2) = -73$

$\therefore -15 - 2n + 2 = -73$

$\therefore -2n = -60$

$\therefore n = 30$

$\therefore$ There are 30 terms in the progression.

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{90} = \frac{30}{2}(-15 - 73) = 15(-88) = -1\ 320$

e) $a = 105$ and $d = -27$

$T_n = a + (n - 1)d = -840$

$\therefore 105 + (n - 1)(-27) = -840$

$\therefore 105 - 27n + 27 = -840$

$\therefore -27n = -972$

$\therefore n = 36$

$\therefore$ There are 36 terms in the progression.

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{90} = \frac{36}{2}(105 - 840) = 18(-735) = -13\ 230$

f) $100 - 4 + 99 - 8 + 98 - 12 + \ldots + 1 - 400$

$= (100 - 4) + (99 - 8) + (98 - 12) + \ldots + (1 - 400)$

$= 96 + 91 + 86 + \ldots + (-399)$

$a = 96$ and $d = -5$

$T_n = a + (n - 1)d = -399$

$\therefore 96 + (n - 1)(-5) = -399$

$\therefore 96 - 5n + 5 = -399$

$\therefore -5n = -500$

$\therefore n = 100$

∴ There are 100 terms in the progression.

$S_n = \frac{n}{2}(a + l)$

$\therefore S_{100} = \frac{100}{2}(96 - 399) = 50(-303) = -15\ 150$

Exercise 4.8

1. $a = 6$ and $d = 2$

a) $T_n = a + (n - 1)d$

$\therefore T_{20} = 6 + (20 - 1)2 = 44$

∴ The 20th pile will contain 44 coins.

b) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{50} = \frac{50}{2}[2(6) + (50 - 1)2] = 25(12 + 98) = 2\ 750$

∴ There will be 2 750 coins in the first 50 piles.

c) $T_n = a + (n - 1)d = 300$

$\therefore 6 + (n - 1)2 = 300$

$\therefore 6 + 2n - 2 = 300$

$\therefore 2n = 296$

$\therefore n = 148$

∴ The 148th pile will contain 300 coins.

2. $a =$ ₦2 500 and $d =$ ₦1 500

a) $T_n = a + (n - 1)d$

$\therefore T_{10} = 2\ 500 + (10 - 1)1\ 500 =$ ₦16 000

∴ The winner will get ₦16 000.

b) $T_n = a + (n - 1)d = 10\ 000$

$\therefore 2\ 500 + (n - 1)1\ 500 = 10\ 000$

$\therefore 2\ 500 + 1\ 500n - 1\ 500 = 10\ 000$

$\therefore 1\ 500n = 9\ 000$

$\therefore n = 6$

∴ The person in 6th place will get ₦10 000.

c) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{10} = \frac{10}{2}[2(2\ 500) + (10 - 1)1\ 500]$

$= 5(5\ 000 + 13\ 500) =$ ₦92 500

∴ The total prize money is ₦92 500.

3. It is easiest to work with this information if we view the number of minutes spent jogging per week as one term in the pattern. So $a = 10 \times 6 = 60$ and $d = 5 \times 6 = 30$.

a) He will jog for 60 minutes in the first week.

b) He will jog for 90 minutes in the second week.

c) $S_n = [2a + (n - 1)d]$

$\therefore S_9 = \frac{9}{2}[2(60) + (9 - 1)30] = \frac{9}{2}(120 + 240) = 1\ 620$

∴ By the end of the 9th week he will have jogged for 1 620 minutes altogether.

4. $d =$ ₦500 and $T_{10} =$ ₦35 000

a) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{10} = \frac{10}{2}[2a + (10 - 1)500] = 35\ 000$

$\therefore 5[2a + (9)500] = 35\ 000$

$\therefore 2a + 4\ 500 = 7\ 000$

$\therefore a =$ ₦1 250

∴ She saved ₦1 250 in the first week.

b) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{52} = \frac{52}{2}[2(1\ 250) + (52 - 1)500]$

$= 26[2\ 500 + (51)500] =$ ₦728 000

∴ She will have saved ₦728 000 by the end of the first year.

5. $a = 180°$ and $d = 180°$

a) There is a constant difference of 180° between successive terms.

b) If the angle sum of a triangle is T_1, then the angle sum of a dodecagon is T_{10}.

$T_n = a + (n - 1)d$

$\therefore T_{10} = 180 + (10 - 1)180 = 1\ 800°$

∴ The sum of the interior angles of a dodecagon is 1 800°.

c) $T_n = a + (n - 1)d$

$\therefore 180 + (n - 1)180 = 5\ 400$

$\therefore 180 + 180n - 180 = 5\ 400$

$\therefore 180n = 5\,400$

$\therefore n = 30$

For each term there are $n + 2$ sides, $\therefore$ a polygon with 32 sides has an angle sum of $5\,400°$.

Exercise 4.9

In this exercise students first need to establish whether there is a common ratio or not.

1. $\frac{5}{1} = 5, \frac{9}{5} = \frac{9}{5}$

This is not a geometric progression, because the ratios are not the same.

2. $\frac{5}{1} = 5, \frac{25}{5} = 5, \frac{125}{25} = 5$

This is a geometric progression. The first term is 1 and the common ratio is 5.

3. $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}, \frac{1}{8} \div \frac{1}{4} = \frac{1}{2}, \frac{1}{16} \div \frac{1}{8} = \frac{1}{2}$

This is a geometric progression. The first term is $\frac{1}{2}$ and the common ratio is $\frac{1}{2}$.

4. $\frac{16}{32} = \frac{1}{2}, \frac{8}{16} = \frac{1}{2}, \frac{4}{8} = \frac{1}{2}$

This is a geometric progression. The first term is 32 and the common ratio is $\frac{1}{2}$.

5. $\frac{4}{1} = 4, \frac{16}{4} = 4, \frac{32}{16} = 2$

This is not a geometric progression, because the ratios are not the same.

6. $\frac{2a}{a} = 2, \frac{3a}{2a} = \frac{3}{2}$

This is not a geometric progression, because the ratios are not the same.

7. $\frac{a^3}{a^2} = a, \frac{a^4}{a^3} = a, \frac{a^5}{a^4} = a$

This is a geometric progression. The first term is a^2 and the common ratio is a.

8. $\frac{36}{216} = \frac{1}{6}, \frac{6}{36} = \frac{1}{6}, \frac{1}{6} = \frac{1}{6}$

This is a geometric progression. The first term is 216 and the common ratio is $\frac{1}{6}$.

9. $\frac{33}{-99} = -\frac{1}{3}, \frac{-11}{33} = -\frac{1}{3}, \frac{\frac{11}{3}}{-11} = -\frac{1}{3}$

This is a geometric progression. The first term is -99 and the common ratio is $-\frac{1}{3}$.

10. $\frac{0.4}{1} = 0.4$, $\frac{0.16}{0.4} = 0.4$

This is a geometric progression. The first term is 1 and the common ratio is 0.4.

Exercise 4.10

1. **a)** 1, 4, 16, 64, 256 **b)** 81, 27, 9, 3, 1

c) 7, 49, 343, 2 401, 16 807 **d)** 625, 125, 25, 5, 1

e) $xy, 2x^2y^2, 4x^3y^3, 8x^4y^4, 16x^5y^5$

2. **a)** 128, 256, 512 **b)** −24, −48, −96

c) −27, 9, −3 **d)** 150, 750, 3 750

e) $\frac{a^5}{16b}, \frac{a^6}{32b}, \frac{a^7}{64b}$

Exercise 4.11

1. **a)** $T_n = ar^{n-1} = 1 \times 7^{n-1} = 7^{n-1}$

b) $T_n = ar^{n-1} = 5 \times (-2)^{n-1}$

c) $T_n = ar^{n-1} = -10 \times 3^{n-1}$

d) $T_n = ar^{n-1} = 32 \times \left(\frac{1}{2}\right)^{n-1} = 2^5 \times (2^{-1})^{n-1} = 2^5 \times 2^{1-n}$
$= 2^{6-n}$

e) $T_n = ar^{n-1} = -9 \times \left(\frac{1}{3}\right)^{n-1} = -(3)^2 \times (3^{-1})^{n-1}$
$= -(3)^2 \times 3^{1-n} = -(3^{3-n})$

f) $T_n = ar^{n-1} = x \times (x^2)^{n-1} = x \times x^{2n-2} = x^{2n-1}$

2. **a)** $a = 1$ and $r = 2$
$\therefore T_n = ar^{n-1} = 1 \times 2^{n-1} = 2^{n-1}$

b) $a = 5$ and $r = 2$
$\therefore T_n = ar^{n-1} = 5 \times 2^{n-1}$

c) $a = 20$ and $r = \frac{3}{2}$
$\therefore T_n = ar^{n-1} = 20 \times \left(\frac{3}{2}\right)^{n-1}$

d) $a = 100$ and $r = \frac{1}{10}$
$\therefore T_n = ar^{n-1} = 100 \times \left(\frac{1}{10}\right)^{n-1} = 10^2 \times (10^{-1})^{n-1}$
$= 10^2 \times 10^{1-n} = 10^{3-n}$

e) $a = 2.5$ and $r = 10$
$\therefore T_n = ar^{n-1} = 2.5 \times 10^{n-1} = \frac{10}{4} \times 10^{n-1} = \frac{10^n}{4}$

f) $a = 1$ and $r = 0.1$
$\therefore T_n = ar^{n-1} = 1 \times 0.1^{n-1} = 0.1^{n-1}$

3. $a = -256$ and $r = -\frac{1}{4}$
$\therefore T_n = ar^{n-1} = -256 \times \left(-\frac{1}{4}\right)^{n-1} = -4^4 \times (-4^{-1})^{n-1}$
$= -4^4 \times (-1)^{n-1} \times 4^{1-n} = (-1)^n \times 4^{5-n}$

Exercise 4.12

1. a) $T_n = 4 \times 5^{n-1}$
$\therefore T_1 = 4 \times 5^{1-1} = 4 \times 5^0 = 4 \times 1 = 4$
$T_n = ar^{n-1}$
$\therefore ar^{n-1} = 4 \times 5^{n-1}$
We know that $a = 4$, so $r = 5$.

b) $T_n = -6 \times 2^n$
$\therefore T_1 = -6 \times 2^{1-1} = -6 \times 2^0 = -6 \times 1 = -6$
$T_n = ar^{n-1}$
$\therefore ar^{n-1} = -6 \times 2^{n-1}$
We know that $a = -6$, so $r = 2$.

c) $T_n = 7 \times (-2)^{n-1}$
$\therefore T_1 = 7 \times (-2)^{1-1} = 7 \times (-2)^0 = 7 \times 1 = 7$
$T_n = ar^{n-1}$
$\therefore ar^{n-1} = 7 \times (-2)^{n-1}$
We know that $a = 7$, so $r = -2$.

d) $T_n = 8^{n-1}$
$\therefore T_1 = 8^{1-1} = 8^0 = 1$
$T_n = ar^{n-1}$
$\therefore ar^{n-1} = 8^{n-1}$
We know that $a = 1$, so $r = 8$.

e) $T_n = (-2.5)^n$
$\therefore T_1 = (-2.5)^1 = -2.5$
$T_n = ar^{n-1}$
$\therefore -2.5r^{n-1} = (-2.5)^n$
$\therefore r^{n-1} = (-2.5)^{n-1}$
$\therefore r = -2.5$

f) $T_n = 10 \times \left(\frac{1}{2}\right)^{n-1}$
$\therefore T_1 = 10 \times \left(\frac{1}{2}\right)^{1-1} = 10 \times \left(\frac{1}{2}\right)^0 = 10 \times 1 = 10$
$T_n = ar^{n-1}$
$\therefore ar^{n-1} = 10 \times \left(\frac{1}{2}\right)^{n-1}$
We know that $a = 10$, so $r = \frac{1}{2}$.

2. **a)** $80 \times \left(\frac{1}{2}\right)^{n-1} = \frac{5}{8}$

$\therefore \left(\frac{1}{2}\right)^{n-1} = \frac{5}{640} = \frac{1}{128} = \left(\frac{1}{2}\right)^{7}$

$\therefore n - 1 = 7$

$\therefore n = 8$

b) $(-2)^{n-1} = -512 = (-2)^9$

$\therefore n - 1 = 9$

$\therefore n = 10$

c) $-6 \times 4^{n-1} = -6\ 144$

$\therefore 4^{n-1} = 1\ 024 = 4^5$

$\therefore n - 1 = 5$

$\therefore n = 6$

d) $1.5 \times 3^{n-1} = 364.5$

$\therefore 3^{n-1} = 243 = 3^5$

$\therefore n - 1 = 5$

$\therefore n = 6$

3. **a)** $T_n = 1.5^n$

$\therefore T_5 = 1.5^5 = 7.59375$

b) $1.5^n = \frac{2\ 187}{128} = \frac{3^7}{2^7} = \left(\frac{3}{2}\right)^7 = 1.5^7$

$\therefore n = 7$

4. **a)** $T_n = ar^{n-1}$

$\therefore T_6 = ar^{6-1} = ar^5$

$\therefore ar^5 = 320$ ①

$T_n = ar^{n-1}$

$\therefore T_{11} = ar^{11-1} = ar^{10}$

$\therefore ar^{10} = 10\ 240$ ②

Divide ② by ①:

$\frac{ar^{10}}{ar^5} = \frac{10\ 240}{320}$

$\therefore r^5 = 32 = 2^5$

$\therefore r = 2$

Substitute 2 for r in ①:

$a(2)^5 = 320$

$\therefore 32a = 320$

$\therefore a = 10$

$\therefore$ The first term is 10 and the common ratio is 2.

b) $T_n = ar^{n-1}$

$\therefore T_4 = ar^{4-1} = ar^3 = 10 \times 2^3 = 80$

c) $T_n = 10 \times 2^{n-1} = 20\,480$
$\therefore 2^{n-1} = 2\,048 = 2^{11}$
$\therefore n - 1 = 11$
$\therefore n = 12$
$\therefore$ There are 12 terms in the progression.

5. $r = \frac{192}{96} = 2$
$T_n = ar^{n-1}$
$\therefore T_8 = a(2)^{8-1} = 96$
$\therefore a(2)^7 = 96$
$\therefore 128a = 96$
$\therefore a = \frac{3}{4}$
$T_n = ar^{n-1}$
$\therefore T_5 = \frac{3}{4}(2)^{5-1} = \frac{3}{4} \times 2^4 = 12$

Exercise 4.13

1. a) $\frac{x}{9} = \frac{81}{x}$
$\therefore x^2 = 729$
$\therefore x = \pm\sqrt{729}$
$\therefore x = \pm 27$

b) $\frac{x}{16} = \frac{1}{x}$
$\therefore x^2 = 16$
$\therefore x = \pm\sqrt{16}$
$\therefore x = \pm 4$

c) $\frac{x}{-20} = \frac{-5}{x}$
$\therefore x^2 = 100$
$\therefore x = \pm\sqrt{100}$
$\therefore x = \pm 10$

d) $\frac{x}{\frac{1}{9}} = \frac{1}{x}$
$\therefore x^2 = \frac{1}{9}$
$\therefore x = \pm\sqrt{\frac{1}{9}}$
$\therefore x = \pm\frac{1}{3}$

e) $\frac{x}{3.5} = \frac{14}{x}$
$\therefore x^2 = 49$
$\therefore x = \pm\sqrt{49}$
$\therefore x = \pm 7$

f) $\frac{x}{\frac{25}{3}} = \frac{\frac{1}{3}}{x}$
$\therefore x^2 = \frac{25}{9}$
$\therefore x = \pm\sqrt{\frac{25}{9}}$
$\therefore x = \pm\frac{5}{3}$

g) $\frac{x}{153} = \frac{17}{x}$
$\therefore x^2 = 2\,601$
$\therefore x = \pm\sqrt{2\,601}$
$\therefore x = \pm 51$

h) $\frac{x}{-0.2} = \frac{-3.2}{x}$
$\therefore x^2 = 0.64$
$\therefore x = \pm\sqrt{0.64}$
$\therefore x = \pm 0.8$

2. a) Let x and y be the two geometric means. So this geometric progression is of the form 4, x, y, 108.
$\therefore 4 \times r^3 = 108$
$\therefore r^3 = \frac{108}{4} = 27$
$\therefore r = 3$

$\therefore x = 4 \times 3 = 12$ and $y = 12 \times 3 = 36$
$\therefore$ The means are 12 and 36.

b) Let x and y be the two geometric means. So this geometric progression is of the form $-337.5, x, y, -0.1$.
$\therefore -337.5 \times r^3 = -0.1$
$\therefore r^3 = \frac{0.1}{337.5}$
$\therefore r = \frac{1}{15}$
$\therefore x = -337.5 \times \frac{1}{15} = -22.5$ and $y = -22.5 \times \frac{1}{15} = -1.5$
$\therefore$ The means are -22.5 and -1.5.

3. Let x, y and z be the three geometric means. So this geometric progression is of the form $22, x, y, z, 111.375$.
$\therefore 22 \times r^4 = 111.375$
$\therefore r^4 = \frac{111.375}{22} = 5.0625$
$\therefore r = \pm 1.5$
$\therefore x = 22 \times 1.5 = 33$, $y = 33 \times 1.5 = 49.5$ and
$z = 49.5 \times 1.5 = 74.25$
OR
$x = 22 \times -1.5 = -33$, $y = -33 \times -1.5 = 49.5$ and
$z = 49.5 \times -1.5 = -74.25$
$\therefore$ The means are 33, 49.5 and 74.25 or -33, 49.5 and -74.25.

Exercise 4.14

1. a) $a = 1$, $r = 2$ and $n = 20$
$S_n = \frac{a(r^n - 1)}{(r - 1)}$ (Use this formula because $r > 1$.)
$\therefore S_{20} = \frac{1(2^{20} - 1)}{(2 - 1)} = 1\ 048\ 575$

b) $a = 6$, $r = -2$ and $n = 20$
$S_n = \frac{a(r^n - 1)}{(r - 1)}$ (Use this formula because $r > 1$.)
$\therefore S_{20} = \frac{6[(-2)^{20} - 1]}{(-2 - 1)} = -2\ 097\ 150$

c) $a = 5$, $r = \frac{1}{5} = 0.2$ and $n = 20$
$S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)
$\therefore S_{20} = \frac{5(1 - 0.2^{20})}{(1 - 0.2)} = 6.25$

d) $a = -3$, $r = \frac{1}{3}$ and $n = 20$
$S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)
$\therefore S_{20} = \frac{-3\left[1 - \left(\frac{1}{3}\right)^{20}\right]}{\left(1 - \frac{1}{3}\right)} = -4.50$

e) $a = 70$, $r = \frac{1}{2}$ and $n = 20$

$S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)

$\therefore S_{20} = \frac{70\left[1 - \left(\frac{1}{2}\right)^{20}\right]}{\left(1 - \frac{1}{2}\right)} = 140.00$

f) $a = 7$, $r = \frac{1}{7}$ and $n = 20$

$S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)

$\therefore S_{20} = \frac{7\left[1 - \left(\frac{1}{7}\right)^{20}\right]}{\left(1 - \frac{1}{7}\right)} = 8.17$

2. $a = -100$ and $r = \frac{1}{2}$

a) $S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)

$\therefore S_5 = \frac{-100\left[1 - \left(\frac{1}{2}\right)^{5}\right]}{\left(1 - \frac{1}{2}\right)} = -193.75$

b) $S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)

$\therefore S_{10} = \frac{-100\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{\left(1 - \frac{1}{2}\right)} = -199.80$

c) $S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)

$\therefore S_{20} = \frac{-100\left[1 - \left(\frac{1}{2}\right)^{20}\right]}{\left(1 - \frac{1}{2}\right)} = -199.90$

d) As the number of terms in the sum increases, the answer gets closer to -200.

Exercise 4.15

1. a) $r = 4$, so $r > 1$. This geometric progression diverges and will not have a sum to infinity.

b) $r = \frac{1}{3}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

c) $r = \frac{1}{6}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

d) $r = -\frac{1}{5}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

e) $r = 6$, so $r > 1$. This geometric progression diverges and will not have a sum to infinity.

f) $r = -\frac{4}{5}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

g) $r = \frac{1}{5}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

h) $r = 1.1$, so $r > 1$. This geometric progression diverges and will not have a sum to infinity.

i) $r = \frac{3}{5}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

j) $r = \frac{1}{10}$, so $-1 < r < 1$. This geometric progression converges and will have a sum to infinity.

2. a), e) and h) will not have a sum to infinity.

b) $a = \frac{1}{4}$ and $r = \frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{3}} = \frac{1}{4} \div \frac{2}{3} = \frac{3}{8}$

c) $a = 216$ and $r = \frac{1}{6}$

$S_\infty = \frac{a}{1-r} = \frac{216}{1-\frac{1}{6}} = 216 \div \frac{5}{6} = \frac{1\,296}{5}$

d) $a = -1.25$ and $r = -\frac{1}{5}$

$S_\infty = \frac{a}{1-r} = \frac{-1.25}{1+\frac{1}{5}} = -1.25 \div \frac{6}{5} = \frac{25}{24}$

f) $a = 100$ and $r = -\frac{4}{5}$

$S_\infty = \frac{a}{1-r} = \frac{100}{1+\frac{4}{5}} = 100 \div \frac{9}{5} = \frac{500}{9}$

g) $a = 5$ and $r = \frac{1}{5}$

$S_\infty = \frac{a}{1-r} = \frac{5}{1-\frac{1}{5}} = 5 \div \frac{4}{5} = \frac{25}{4}$

i) $a = 50$ and $r = \frac{3}{5}$

$S_\infty = \frac{a}{1-r} = \frac{50}{1-\frac{3}{5}} = 50 \div \frac{2}{5} = 125$

j) $a = 1$ and $r = \frac{1}{10}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = 1 \div \frac{9}{10} = \frac{10}{9}$

3. $S_\infty = \frac{a}{1-r}$

$\therefore a = 5[1 - (-0.2)]$

$= 5(1.2)$

$= 6$

The first three terms are 6, −1.2 and 0.24.

Exercise 4.16

1. $a = 2$, $r = 2$ and $n = \frac{30}{3} = 10$

$S_n = \frac{a(r^n - 1)}{(r-1)}$ (Use this formula because $r > 1$.)

$\therefore S_{10} = \frac{2(2^{10} - 1)}{(2-1)} = 2\,046$

∴ The size of the colony after 30 weeks was 2 046.

2. $a = 982$, $r = 1.09$ and $n = 7$
($r = \frac{109}{100} = 1.09$)
$T_n = ar^{n-1}$
$\therefore T_7 = 982(1.09)^{7-1} \approx 1\,647$
$\therefore$ Approximately 1 647 people were infected by the end of the second week.

3. Students' reasoning may differ. One line of reasoning is as follows: There was the initial drop of 5 m and then a number of bounces. The total distance travelled was $2S_\infty - 5$, where $a = 5$ and $r = \frac{2}{3}$. We multiply by 2, because on each bounce the ball travelled up and down. Also we must subtract 5 m, because the first drop of 5 m was in a downwards direction only.
$$\begin{aligned}\therefore \text{The total distance} &= 2\left(\tfrac{a}{1-r}\right) - 5\\ &= 2 \times 5 \div \left(1 - \tfrac{2}{3}\right) - 5\\ &= 2 \times 5 \div \tfrac{1}{3} - 5\\ &= 30 - 5\\ &= 25 \text{ m}\end{aligned}$$

4. $a = 0.46$ and $r = 1.1$
($r = \frac{110}{100} = 1.1$)
$T_n = ar^{n-1} = 0.8$
$\therefore 0.46(1.1)^{n-1} = 0.8$
$\therefore 1.1^{n-1} = \frac{0.8}{0.46} = 1.739\ldots$
Using trial-and-improvement, your students will find that:
$1.1^5 = 1.61\ldots$, which is smaller than $1.739\ldots$, and
$1.1^6 = 1.77\ldots$, which is greater than $1.739\ldots$.
$\therefore n - 1 = 6$
$\therefore n = 7$
$\therefore$ He should achieve his goal when he writes the 7th test.

Alternative method, using logs:
$1.1^{n-1} = \frac{0.8}{0.46}$
$\therefore \log 1.1^{n-1} = \log \frac{0.8}{0.46}$
$\therefore (n-1)(\log 1.1) = \log \frac{0.8}{0.46}$
$\therefore n - 1 = \log \frac{0.8}{0.46} \div \log 1.1 = 5.80615\ldots$
$\therefore n = 6.80615\ldots$
$\therefore$ He should achieve his goal when he writes the 7th test.

5. **a)** In the third year, the tree will grow by $\frac{1}{4}$ of 30 cm = 7.5 cm. The height of the tree by the end of the third year will be 120 cm + 30 cm + 7.5 cm = 157.5 cm.

b) This forms a geometric progression with $a = 120$ and $r = \frac{1}{4}$. Even if the tree were to grow indefinitely, the maximum height of the tree is:

$S_\infty = \frac{a}{1 - r} = \frac{120}{1 - \frac{1}{4}} = 120 \div \frac{3}{4} = 160$ cm

$\therefore$ The tree can never grow beyond a height of 160 cm.

6. $a = 650$ mg, $r = 0.5$ and $n = \frac{14}{2} = 7$

$T_n = ar^{n-1}$

$\therefore T_7 = 650(0.5)^{7-1} = 10.16$ mg

$\therefore$ After two weeks there will be 10.16 mg of radioactive material in the ore sample.

Assess your progress

1. **a)** $9 - 3 = 6,\ 15 - 9 = 6,\ 21 - 15 = 6$

This is an arithmetic progression. The common difference is 6.

b) This is neither.

c) $\frac{15}{5} = 3,\ \frac{45}{15} = 3,\ \frac{135}{45} = 3$

This is a geometric progression. The common ratio is 3.

d) $14 - 16 = -2,\ 12 - 14 = -2,\ 10 - 12 = -2$

This is an arithmetic progression. The common difference is −2.

e) This is neither.

f) $\frac{-5}{-10} = \frac{1}{2},\ \frac{-2.5}{-5} = \frac{1}{2},\ \frac{-1.25}{-2.5} = \frac{1}{2}$

This is a geometric progression. The common ratio is $\frac{1}{2}$.

2. **a)** $a = 3$ and $d = 6$

$\therefore T_n = a + (n - 1)d = 3 + (n - 1)(6) = 3 + 6n - 6$
$= 6n - 3$

c) $a = 5$ and $r = 3$

$\therefore T_n = ar^{n-1} = 5 \times 3^{n-1}$

d) $a = 16$ and $d = -2$

$\therefore T_n = a + (n - 1)d = 16 + (n - 1)(-2) = 16 - 2n + 2$
$= -2n + 18$

f) $a = -10$ and $r = \frac{1}{2}$

$\therefore T_n = ar^{n-1}$

$= -10\left(\frac{1}{2}\right)^{n-1}$

$= -10 \times 2^{1-n}$

3. **a)** $a = 1$ and $T_4 = 64$

$T_4 = 64$

$\therefore 1 + (4 - 1)d = 64$

$\therefore 1 + 3d = 64$

$\therefore 3d = 63$

$\therefore d = 21$

$\therefore T_2 = 1 + 21 = 22$ and $T_3 = 22 + 21 = 43$

$\therefore$ The means are 22 and 43.

b) $a = 1$ and $T_4 = 64$

$T_4 = 64$

$\therefore 1(r)^{4-1} = 64$

$\therefore r^3 = 64$

$\therefore r = 4$

$\therefore T_2 = 1 \times 4 = 4$ and $T_3 = 4 \times 4 = 16$

$\therefore$ The means are 4 and 16.

4. **a)** $T_n = 8n - 3$

$\therefore T_1 = 8(1) - 3 = 5$

b) $T_2 = 8(2) - 3 = 13$

$\therefore d = 13 - 5 = 8$

c) $S_n = \frac{n}{2}[2a + (n - 1)d]$

$\therefore S_{200} = \frac{200}{2}[2(5) + (200 - 1)(8)] = 100[10 + (199)(8)]$

$= 160\ 200$

5. **a)** $r = \frac{T_6}{T_5} = \frac{2}{10} = \frac{1}{5}$

b) $T_3 = T_5 \div r^2 = 10 \div \frac{1}{25} = 250$

c) $T_1 = T_3 \div r^2 = 250 \div \frac{1}{25} = 6\ 250$

$S_n = \frac{a(1 - r^n)}{(1 - r)}$ (Use this formula because $-1 < r < 1$.)

$\therefore S_6 = \frac{6\ 250\left[1 - \left(\frac{1}{5}\right)^6\right]}{\left(1 - \frac{1}{5}\right)} = 7\ 812$

d) $S_\infty = \frac{a}{1 - r} = \frac{6\ 250}{1 - \frac{1}{5}} = 6\ 250 \div \frac{4}{5} = 7\ 812.5$

6. First calculate the sum of the first 10 000 integers:
$a = 1$, $d = 1$ and $n = 10\,000$
$S_n = \frac{n}{2}(a + l)$
$\therefore S_{10\,000} = \frac{10\,000}{2}(1 + 10\,000) = 5\,000(10\,001)$
$= 50\,005\,000$
Now calculate the sum of all the multiples of 5:
$a = 5$, $d = 5$ and $n = \frac{10\,000}{5} = 2\,000$
$S_n = \frac{n}{2}(a + l)$
$\therefore S_{2\,000} = \frac{2\,000}{2}(5 + 10\,000) = 1\,000(10\,005) = 10\,005\,000$
$50\,005\,000 - 10\,005\,000 = 40\,000\,000$
$\therefore$ The sum of the remaining numbers is 40 000 000.

7. The 25th triangular number is the sum of the first 25 integers. This is an arithmetic progression with $a = 1$, $d = 1$ and $n = 25$.
$S_n = \frac{n}{2}(a + l)$
$\therefore S_{25} = \frac{25}{2}(1 + 25) = \frac{25}{2}(26) = 325$
$\therefore$ The 25th triangular number is 325.

8. $a = 900$, $r = 2$ and $n = 24 \times 2 = 48$
$T_n = ar^{n-1}$
$\therefore T_{48} = ar^{48-1} = ar^{47} = 900 \times 2^{47} = 1.266637395 \times 10^{17}$
$\therefore$ There will be $1.266637395 \times 10^{17}$ bacteria after 24 hours.

TOPIC 5 Quadratic equations

Introduction

In this topic we begin by revising different factorisation methods with the students:

- highest common factor
- difference of two squares
- trinomials
- perfect squares.

We then revise the method for completing a perfect square trinomial expression, and use this method to derive the quadratic formula for solving quadratic equations.

The next section deals with the derivation of the quadratic formula, and how to use the formula.

Then we look at how the quadratic formula gives the two roots of a quadratic equation, and how we can use the sum of roots and product of roots to create a quadratic equation. We also study the method of writing down a quadratic equation if the given roots are x_1 and x_2.

Finally we use quadratic equations to solve word problems. In this section, we look at area, number and distance–speed–time problems.

Common difficulties

Remind students that $(+)^2 = +$ and $(-)^2 = +$. This is a very important mathematical fact to remember in factorisation.

Students must be clear on the difference between an expression that they need to simplify or factorise, and an equation that they need to solve.

Remind students that when factorising they should always look for a common factor first.

Preparation

Prepare a chart showing the different methods of factorisation.

Have a list of at least the first twelve square numbers that the students really need to know: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144.

Make a chart of the factors of the numbers used many times when factorising equations, for example:

Factors of 12: 1, 2, 3, 4, 6, 12

Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

Introduction for students

Revise basic factorisation methods with a few examples of each type. Focus on perfect squares. Explain how to add a constant term to an expression to form a perfect square.

Explain how important it is to be able to factorise an expression before we can start solving equations.

Remind students about completing the square and using the quadratic formula.

Answers

Exercise 5.1

1. **a)** $8x(2a - 4y + ay)$ **b)** $2a(2a - 1)$
c) $25ab(a + 2b)$ **d)** $(4x - 5y)(4x + 5y)$
e) $(3y - 2)(y - 3)$ **f)** $(9 - a)(9 + a)$
g) $4(x + 5)(x - 3)$ **h)** $(a + 4)^2$

2. **a)** $k = 16$ **b)** $k = 100$ **c)** $k = 9$ **d)** $k = 64$
e) $k = 36$ **f)** $k = 49$ **g)** $k = 1$ **h)** $k = 9$

3. **a)** $(x - 4)^2$ **b)** $(x + 10)^2$ **c)** $(x - 3)^2$
d) $(x + 8)^2$ **e)** $(x - 6)^2$ **f)** $(x + 7)^2$
g) $(3x + 1)^2$ **h)** $(4x + 3)^2$

Exercise 5.2

1. **a)** $x = \pm\frac{1}{2}$ **b)** $x = 2$ or $x = 3$
c) $x = 0$ or $x = 20$ **d)** $x = 0$ or $x = 3$
e) $x = 3$ or $x = 4$ **f)** $x = -7$
g) $x = \pm\frac{1}{8}$ **h)** $x = 3$

2. **a)** $x = 5$ **b)** $x = 9$ **c)** $x = 8$
d) $x = 2$ **e)** $x = -\frac{5}{2}$ **f)** $x = 5$
g) $x = -\frac{1}{3}$ **h)** $x = \frac{6}{5}$

3. a) $k = 36$ b) $k = 81$ c) $k = 121$
d) $k = 196$ e) $k = 49$ f) $k = 8$
g) $k = 1$ h) 9

4. a) $x = 6$ b) $x = -9$ c) $x = 11$
d) $x = 14$ e) $x = -7$ f) $x = 2$
g) $x = \frac{1}{3}$ h) $x = -\frac{3}{2}$

Exercise 5.3

1. $x = 2 \pm \sqrt{11}$
2. $x = -4 \pm \sqrt{21}$
3. $x = 3 \pm \sqrt{21}$
4. $x = \frac{-3 \pm \sqrt{73}}{4}$
5. $x = \frac{-7 \pm \sqrt{93}}{2}$
6. $x = \frac{2 \pm \sqrt{2}}{3}$
7. $x = \frac{-1 \pm \sqrt{5}}{2}$
8. $x = \frac{4 \pm \sqrt{14}}{3}$

Exercise 5.4

1. a) $x = \frac{3 \pm \sqrt{13}}{2}$ b) $x = \frac{10 \pm \sqrt{148}}{-6}$
c) $x = \frac{-2 \pm \sqrt{8}}{2}$ d) $x = \frac{4 \pm \sqrt{32}}{2}$
e) $x = \frac{-2 \pm \sqrt{52}}{6}$ f) $x = \frac{b \pm \sqrt{b^2 + 4ac}}{2a}$

2. a) $x = 2.18$ or $x = 0.15$ b) $x = -0.68$ or $x = 0.88$
c) $x = 0.59$ or $x = 3.41$ d) $x = 0.77$ or $x = -0.43$
e) $x = 1.28$ or $x = -0.78$ f) $x = 1.77$ or $x = 0.57$

3. $\left(x - \frac{7}{2}\right)^2 + \frac{19}{4}$; the minimum value is $\frac{19}{4}$

4. $p = -\frac{5}{2}$

5. $p = \pm 12$

Exercise 5.5

1. a) $x^2 - x - 6 = 0$ b) $x^2 - 13x + 42 = 0$
c) $4x^2 - 9x + 2 = 0$ d) $3x^2 - 7x + 2 = 0$
e) $x^2 + 4x + 1 = 0$ f) $x^2 - 6x - 2 = 0$
g) $x^2 - k^2 = 0$ h) $kx^2 - (1 + k^2)x + k = 0$

2. a) $p = -5$ or $p = 11$ b) $k = -\frac{1}{2}$

3. $3x^2 + 5x - 12 = 0$

4. a) $k = -6$; other root $= -\frac{3}{2}$ b) $k = -2$; other root $= 3$
c) $a = -7$; other root $= 2$ d) $p = 2$; other root $= 2$

5. $x = 1.16$ or $x = -5.16$

Exercise 5.6

1. 8 hours
2. 24
3. 40
4. 15 km/h
5. 11 and 13, or −11 and −13
6. 10 hrs

Assess your progress

1. a) $(2x - 5y)(2x + 5y)$ b) $(a - 1)^2$
   c) $(x - 5)(x + 4)$ d) $(2x - 1)(x - 3)$
   e) $(y - 5)^2$ f) $(x + 6)(x - 5)$

2. a) $k = 81$ b) $k = 36$
   c) $k = \frac{9}{4}$ d) $k = 4$

3. a) $x = \frac{-5 \pm \sqrt{37}}{2}$ b) $x = \frac{3 \pm \sqrt{19}}{2}$

4. $k = \pm 30$

5. a) $x^2 + 2x - 15 = 0$ b) $x^2 + 2x - 1 = 0$
   c) $x^2 - 6x + 4 = 0$

6. a) $x = 4$ b) $x = \frac{1}{4}$ or $x = 2$
   c) $x = -2 \pm 3\sqrt{2}$; 2.24 or −6.24
   d) $x = -0.18$ or $x = -2.82$

7. $k = 3$; $x = -\frac{1}{3}$

8. $x = 80$

9. One windmill takes 20 hours and the other windmill takes 30 hours.

10. One side is 7 cm and the other side is 24 cm.

11. The number is 3 or −12.

TOPIC 6 Simultaneous equations

Introduction

We begin this topic by solving simultaneous linear equations using elimination of a variable first, and then the substitution method, which is more useful when both equations are not linear.

Then we solve simultaneous equations containing indices.

The section that follows deals with linear and non-linear equations and functions. We start by revising straight line graphs, and then find the point of intersection of two lines graphically and algebraically. Then we move on to finding the point(s) of intersection of a straight line and a quadratic graph by looking at the graph or by creating simultaneous equations.

Finally we apply the method of solving simultaneous equations to real-life problems involving area, ages and numbers.

Common difficulties

One of the main problems that occurs when solving simultaneous equations is making a careless mistake. For example, when subtracting $-5x$ from $4x$, students need to remember to apply the rules for subtraction carefully: $4x - (-5x) = 9x$.

Students need to get used to the substitution method as that is the method they will have to use when solving simultaneous equations that are not both linear.

Students should draw rough sketches of graphs whenever possible to check that the answers to the simultaneous equations make logical sense.

When solving exponential equations, students need to remember that when multiplying powers with the same bases, the base stays the same and the rules apply to the indices. For example $3^y \times 3^2$ is equal to 3^{y+2} and **not** 9^{y+2}!

Students often solve for one of the variables and then forget to substitute to find the value of the other variable.

Advise them that when answering a question in an exercise, test or exam, they should remember to go back and read the question to make sure that the answer is complete.

When solving an equation containing algebraic fractions, students must remember to check whether their answers are valid. If the answer gives zero in the denominator then that answer is undefined and must be rejected.

Preparation

Have graph paper, pencils, rulers, and if possible mathematical sets available for drawing straight lines.

Prepare charts illustrating:

- a Cartesian plane showing the x and y axes and the origin
- two intersecting lines
- a line intersecting a quadratic graph at two points
- a tangent touching a quadratic graph, to show that there will be only one solution with two equal roots.

Introduction for students

Explain that when there are two variables for which values need to be found, we need two equations that need to be solve simultaneously, which means at the same time.

Show the students the elimination method which can be used for solving two linear equations.

Then show students the substitution method and emphasise the importance of being able to substitute into an equation. They need to get an equation that has only one variable to solve. The substitution method is the one that they will need to know how to use when the equations are not both linear.

Answers

Exercise 6.1

1. **a)** $x = 8; y = 6$ **b)** $x = -\frac{4}{3}; y = \frac{1}{3}$
c) $x = 2; y = 1$ **d)** $x = 3; y = 1$
e) $x = 5; y = 3$ **f)** $x = 5; y = 6$

2. **a)** $x = -21; y = 16$ **b)** $x = 4; y = 1$
c) $x = 2; y = 1$ **d)** $x = 2; y = 4$
e) $x = 2; y = 1$ **f)** $x = 4; y = 2$

3. **a)** $x = 3; y = 1$ **b)** $x = 3; y = -7$
c) $x = -9; y = 17$ **d)** $x = 3; y = 4$
e) $x = \frac{1}{3}; y = -2$ **f)** $x = 3; y = 1$

4. $a = 4; b = 2$

Exercise 6.2

1. **a)** $x = -2; y = 1$ **b)** $x = 3; y = -1$
c) $x = -3; y = 2$ **d)** $x = 5; y = -2$
e) $x = 2; y = 1$ **f)** $x = \frac{1}{2}; y = -3$

2. **a)** $x = 5; y = 2$ **b)** $x = 1; y = 2$
c) $x = 3; y = 1$ **d)** $x = -2; y = 1$
e) $x = 4; y = 3$ **f)** $x = 5; y = -2$

3. **a)** $a = 1; b = -2$ **b)** $a = 2; b = -3$

Exercise 6.3

1. $x = -\frac{3}{2}; y = 3$
2. $x = 1; y = \frac{5}{2}$
3. $x = 2; y = -\frac{1}{2}$
4. (1; 4) or (9; 36)
5. $x = 2; y = -1$
6. $x = 3; y = 1$
7. $x = 2; y = 4$
8. $x = 2; y = 0$
9. (2; 4) or $\left(-\frac{22}{5}; \frac{4}{5}\right)$
10. $x = 3; y = -1$

Exercise 6.4

1. **a)** $y = -4x$ **b)** $y = -3x + 6$
c) $y = 2x + 4$ **d)** $y = -\frac{3}{4}x + 3$
e) $y = x - 1$ **f)** $y = 3x + 2$

2. **a)** (2; 1) **b)** (4; 22) **c)** (−2; −1)

Exercise 6.5

1. **a)** $x = 2; y = 2$ **b)** (−1; 4); (4; −1)
c) (−2; −4); (4; 2) **d)** (−5; 8); (8; −5)
e) $\left(-\frac{2}{3}; -\frac{16}{3}\right)$; (3; 13) **f)** (−2; −4); (1; 5)

2. **a)** (−3; −7); (2; 8) **b)** (1; 0); (−5; 6)
c) $\left(3; \frac{3}{2}\right)$; $\left(-1; \frac{5}{2}\right)$ **d)** (1; −3); (7; −1)

e) $\left(\frac{2}{3}; \frac{2}{3}\right); \left(\frac{1}{2}; 1\right)$ f) (4; 3); (−1; −2)

3. a) (20; 8); (12; 4) b) $\left(-3; \frac{1}{2}\right)$; (4; 4)

4. a) (5; 1); (1; −1) b) (0; 1); $\left(\frac{5}{3}; \frac{11}{6}\right)$
c) (3; 1); (2; 4) d) (1; −3); (4; 3)

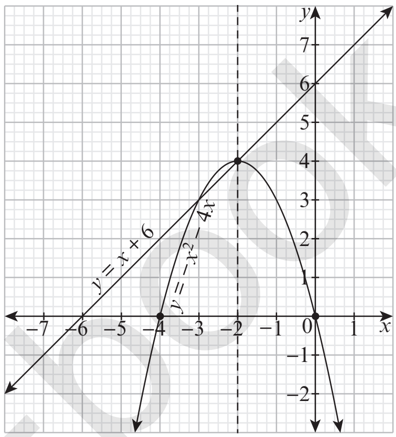
Exercise 6.6

1. (−6; 27); (3; 0) **2.** (−2; 9); (1; 0)

3. (−2; 0); $\left(\frac{3}{2}; \frac{7}{2}\right)$ **4.** AB = 2

5. A(0; 9), B(3; 0), C(−3; 0), D(0; 3), F$\left(\frac{3}{2}; \frac{9}{2}\right)$, G(2; 5)

6. (−2; 4); (−3; 3)



Exercise 6.7

1. 6 m × 10 m
2. length = 32 m and breadth = 8 m
3. base = 8 m and height = 2 m
4. Lewa's age now is 42 years.
5. Ifede is 13 and her brother is 5.
6. 27 and 35
7. 7 and 17
8. 78 or 87
9. 23 and 32
10. 38
11. Fifteen ₦100 notes and five ₦500 notes.

Assess your progress

1. **a)** $a = 6; b = -1$ **b)** $x = 5; y = -3$

2. **a)** $(2; -6); (5; 6)$ **b)** $\left(-4; -\frac{5}{3}\right); (3; 10)$
c) $(0; -8); (4; 0)$

3. $A\left(\frac{3}{2}; 3\right)$

4. $P\left(\frac{5}{2}; \frac{11}{2}\right)$

5. The son is 3 years old.

6. 35 and 25

7. There are 24 girls in the class.

8. 37 and 29

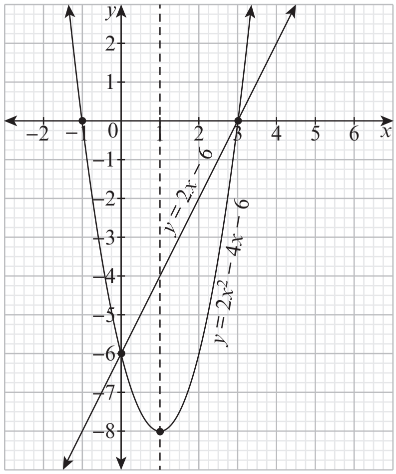
9. 5 m × 4 m

10. There are 15 ₦20 notes in the money bag.

11. There are 20 ₦100 notes.

12. An exercise book costs ₦250 and a pen costs ₦125.

13. Straight line: $y = 2x - 6$
y intercept: $y = -6$
x intercept: $x = 3$
Quadratic graph: $y = 2x^2 - 4x - 6$
Arms of the quadratic graph go up ($a > 0$), so the graph will have a minimum value.
y intercept: $y = -6$
x intercepts: $x = 3$ and $x = -1$
Line of symmetry: $x = -\frac{-4}{4} = 1$
Minimum value: $y = 2(1)^2 - (4)(1) - 6 = -8$
Turning point: $(1; -8)$
Simultaneous equation:
$2x^2 - 4x - 6 = 2x - 6$
$\therefore 2x^2 - 6x = 0$
$\therefore x = 0$ or $x = 3$
Solution: $(0; -6)$ and $(3; 0)$
Points of intersection: $(0; -6)$ and $(3; 0)$



14. Straight line: $y = 2x + 2$

y intercept: $y = 2$

x intercept: $x = -1$

Quadratic graph: $y = -x^2 + 2x + 3$

Arms of the quadratic graph go down ($a < 0$), so the graph will have a maximum value.

y intercept: $y = 3$

x intercepts: $x = 3$ and $x = -1$

Line of symmetry: $x = -\frac{2}{-2} = 1$

Maximum value: $y = -(1)^2 + (2)(1) + 3 = 4$

Turning point: $(1; 4)$

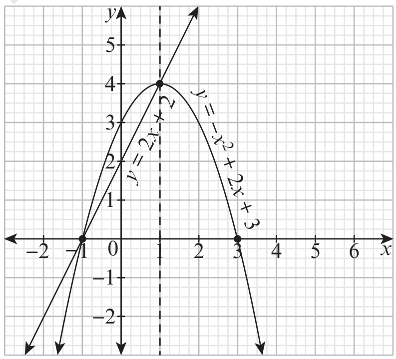
Simultaneous equation: $-x^2 + 2x + 3 = 2x + 2$

$\therefore -x^2 + 1 = 0$

$\therefore x = 1$ or $x = -1$

Solution: $(1; 4)$ and $(-1; 0)$

Points of intersection = $(1; 4)$ and $(-1; 0)$



TOPIC 7 Gradient of a curve

Introduction

We begin by revising straight lines, and finding the equation of a straight line. Then we focus on the gradient of a straight line.

- We look at an equation in standard form $y = mx + c$ and also in general form $ax + by + c = 0$.
- We notice the difference between a positive gradient that results in an increasing graph and a negative gradient that results in a decreasing graph.
- We calculate the gradient of a line using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, and look at the gradients that result in parallel lines and perpendicular lines: parallel lines have equal gradients, and perpendicular lines have gradients that give the product of -1.

We then draw straight lines using the dual intercept method. We note the horizontal line $y = k$ which has a zero gradient, and the vertical line $x = k$ which has an infinite gradient, so the gradient is undefined.

We find the equation of a straight line given:

- the gradient and a point on the line
- the y intercept and a point
- two points.

We move on to working out the gradient of a curve at a point, which is equal to the gradient of the tangent line at that point. We see that the gradient of a tangent at a turning point is zero as the tangent is a horizontal line.

Common difficulties

Students sometimes put the difference between the x values in the numerator of the gradient equation. They need to learn the formula and remember that it is the difference between the y coordinates of the points that forms the numerator: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

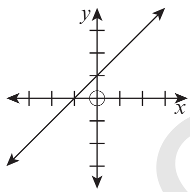
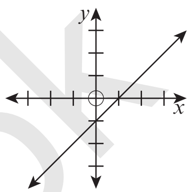
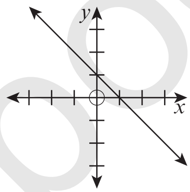
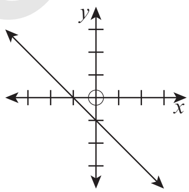
Students need to be consistent in the way they substitute the points into the formula. If they substitute $\frac{y_2 - y_1}{x_2 - x_2}$, then the gradient will be incorrect.

When working out the gradient of a tangent to a point, students should use points where blocks on graph paper can be counted easily to find the gradient.

Preparation

Provide graph paper for drawing accurate graphs and for finding the gradient of a tangent to a curve. Have rulers, pencils and erasers available.

Prepare a chart of positive and negative gradients and intercepts on the *y* axis, as shown below.

	$c > 0$	$c < 0$
$m > 0$		
$m < 0$		

Introduction for students

Revise straight lines and quadratic graphs studies in previous topics and years.

Explain the implications of positive and negative gradients and intercepts on the *y* axis.

Discuss the gradients of horizontal and vertical lines, in other words lines that are parallel to the axes.

Explain that it is only possible to draw in a tangent line roughly until students have learnt calculus, which is needed to calculate a tangent line accurately.

Answers

Exercise 7.1

1. **a)** $m = \frac{2}{3}$ **b)** $m = -4$
c) $m = \frac{1}{3}$ **d)** $m = \frac{1}{2}$

2. **a)** $m = 4$ **b)** $m = 4$
c) $m = -\frac{3}{4}$ **d)** $m = 3$

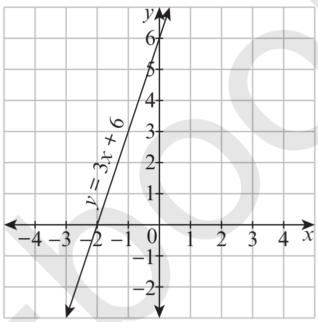
3. **a)** parallel **b)** perpendicular
c) parallel **d)** perpendicular

4. **a)** $m = \frac{6}{8} = \frac{3}{4}$ **b)** $m = \frac{3}{4}$

5. **a)** $m = -\frac{4}{8} = -\frac{1}{2}$ **b)** $m = 2$

Exercise 7.2

1.



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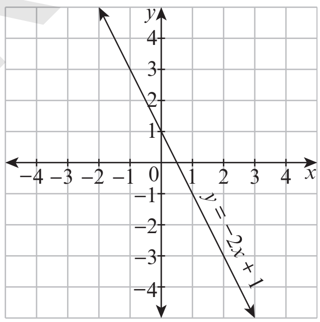

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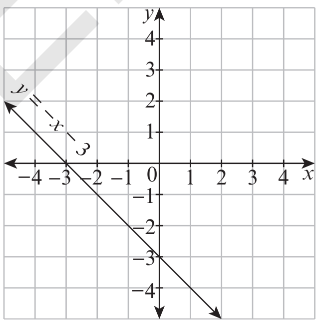

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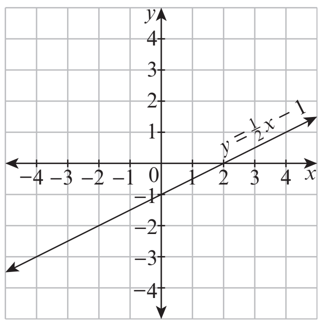

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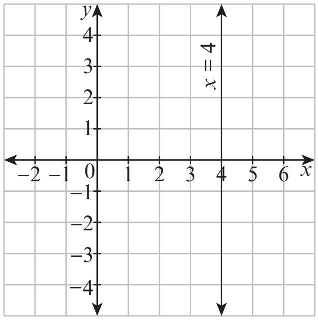

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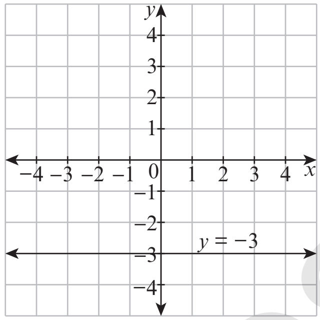

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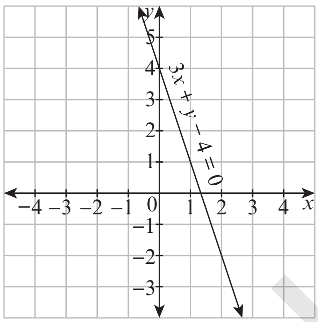

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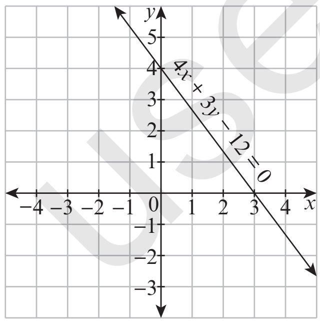
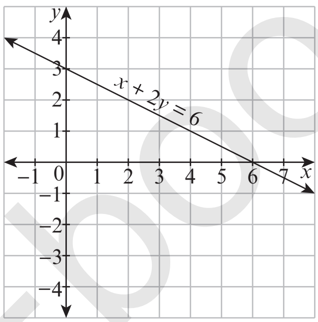


9.



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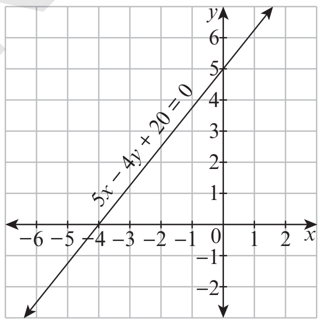


Exercise 7.3

1. **a)** $y = 4x - 1$ **b)** $y = 3x + 3$

c) $y = -2x + 4$ **d)** $y = -\frac{1}{5}x + 1$

2. **a)** $y = 3x + 6$ **b)** $y = -\frac{1}{2}x - 2$

c) $y = -2x + 3$ **d)** $y = -2x + 4$

3. **a)** $y = -\frac{3}{4}x + \frac{11}{2}$ **b)** $y = \frac{1}{2}x + 4\frac{1}{2}$

c) $y = 1$ **d)** $x = 6$

4. **a)** $y = -\frac{1}{3}x$ **b)** $y = \frac{3}{4}x + \frac{3}{2}$

c) $y = -\frac{2}{3}x + 2$ **d)** $y = \frac{2}{3}x - 2$

5. **a)** $y = -4$ **b)** $y = 3$ **c)** $x = -1$

6. $MN = -\frac{2}{3}x$
$RS = \frac{3}{2}x - 4$
$PQ = \frac{3}{2}x - 6$

Exercise 7.4

1. **a)** $m = 0$ **b)** $x = 0$ (the y axis)

2. $m = 2$

3. **a)** $m = 6$ **b)** $y = x^2 - 8x + 12$

4. **a)** $m = -2$ **b)** $m = 2$

5. **a)** $y = -2x + 5$
b) $-2x + 5 = -x^2 + 4$
$x^2 - 2x + 1 = (x - 1)^2$
$x = 1; y = 3$
c) $m = -2$

Assess your progress

1. **a)** $m = -\frac{2}{3}$ **b)** $m = -\frac{3}{4}$
c) $m = 0$ **d)** m = undefined

2. **a)** $m = -\frac{3}{2}$ **b)** $m = 1$
c) $m = -\frac{1}{3}$ **d)** $m = 0$

3. **a)** $y = 6x - 10$ **b)** $y = 2x - 3$
c) $y = -1$ **d)** $y = -\frac{1}{2}x + 6$

4. **a)** $y = 2x - 2$
b) $y = -\frac{5}{2}x + 5$

5. **a)** gradient of AB $= -7$ **b)** y intercept of AB: $c = 19$
c) AB: $y = -7x + 19$
d) line parallel to AB: $y = -7x$
e) line perpendicular to AB: $y = \frac{1}{7}x + \frac{34}{7}$

6. **a)** $y = 2$ **b)** $y = -1$ **c)** $y = 2$

7. $m = 6$

8. $m = 4$

TOPIC 8 Linear inequalities

Introduction

Start by revising how to solve linear inequalities in one variable and illustrate the solution on a number line. Then move on to solving compound linear inequalities in one variable, linear inequalities in two variables and simultaneous linear inequalities.

In the next section introduce students to linear programming for the first time. Many of them will not have heard the terms before and may think it has something to do with computers. Explain to them that linear programming is a method of solving a set of linear inequalities, and that it is a very useful tool for solving many real-life problems.

Common difficulties in this topic

Students often find the inequality symbols confusing, so revise these with them.

You will need to remind your students that if they multiply or divide an inequality by a negative number, the inequality symbol must be changed around.

Students sometimes get confused when shading the feasible region and shade the wrong polygon. This happens when they interpret the inequalities incorrectly. Remediate this problem as it occurs. If students use arrows to indicate the correct direction once they have drawn each inequality, this can help them to identify the feasible region correctly.

It is important that your students draw their inequalities very accurately.

Preparation

A good supply of graph paper is ideal. However, if you do not have access to graph paper, your students will have to draw the axes and then use a ruler to mark off the scales carefully.

You could prepare a chart of a few examples using the four different inequality symbols and representations of these on number lines. Vary the direction of the arrows, and use a mixture of solid and open dots.

Introduction for students

Briefly revise the idea of an inequality with your class. Remind students that a linear equation contains an = sign and has a single solution, but an inequality contains an inequality symbol (<, >, ≤ or ≥) and has many different solutions. For example, $x = 4$ has exactly one solution, but $x < 4$ has an infinite number of solutions.

Write the symbols <, >, ≤ and ≥ on the board and ask your students if they can remember what they mean. Remediate any confusion that your students may have about these symbols before starting to teach this topic.

Use examples of number lines from your chart or on the board, and ask volunteers to describe each inequality in words.

Answers

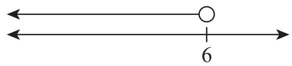
Exercise 8.1

1. a) $5x > 10$

$\therefore x > 2$

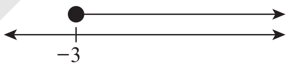
b) $x - 4 < 2$

$\therefore x < 6$



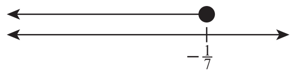
c) $-x \leq 3$

$\therefore x \geq -3$



d) $-7x \geq 1$

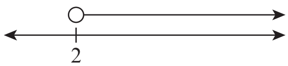
$\therefore x \leq -\frac{1}{7}$



e) $8x + 3 > 19$

$\therefore 8x > 16$

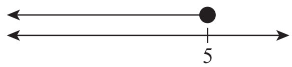
$\therefore x > 2$



f) $6 - 2x \geq -4$

$\therefore -2x \geq -10$

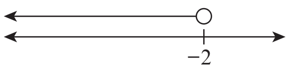
$\therefore x \leq 5$



g) $3x + 15 < 5 - 2x$

$\therefore 5x < -10$

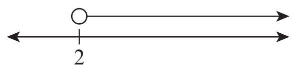
$\therefore x < -2$



h) $7x - 9 > 3x - 1$

$\therefore 4x > 8$

$\therefore x > 2$

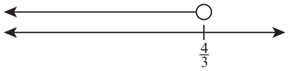


2. **a)** $3(x + 1) < 7$

$\therefore 3x + 3 < 7$

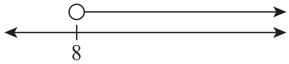
$\therefore 3x < 4$

$\therefore x < \frac{4}{3}$



b) $\frac{x}{4} > 2$

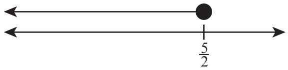
$\therefore x > 8$



c) $-\frac{2x}{5} \geq -1$

$\therefore -2x \geq -5$

$\therefore x \leq \frac{5}{2}$



d) $x \leq \frac{5x}{3} + 4$

$\therefore 3x \leq 5x + 12$

$\therefore -2x \leq 12$

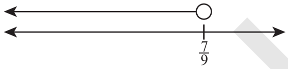
$\therefore x \geq -6$

e) $4(3 - x) > 5(1 + x)$

$\therefore 12 - 4x > 5 + 5x$

$\therefore -9x > -7$

$\therefore x < \frac{7}{9}$



f) $\frac{x}{4} - \frac{x}{5} < -3$

$\therefore 5x - 4x < -60$

$\therefore x < -60$

g) $2(4x - 3) \geq x + 8$

$\therefore 8x - 6 \geq x + 8$

$\therefore 7x \geq 14$

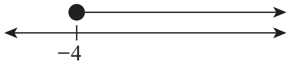
$\therefore x \geq 2$

h) $x \leq 4\left(\frac{x}{2} + 1\right)$

$\therefore x \leq 2x + 4$

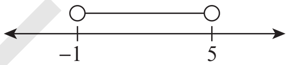
$\therefore -x \leq 4$

$\therefore x \geq -4$



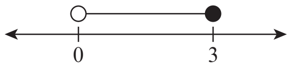
Exercise 8.2

1. **a)** $-1 < x < 5$



b) $0 < 3x \leq 9$

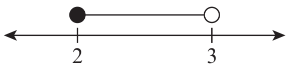
$\therefore 0 < x \leq 3$



c) $5 \leq 2x + 1 < 7$

$\therefore 4 \leq 2x < 6$

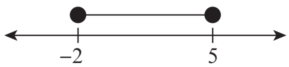
$\therefore 2 \leq x < 3$



d) $-4 \leq 2 + 3x \leq 17$

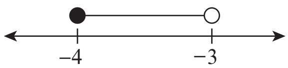
$\therefore -6 \leq 3x \leq 15$

$\therefore -2 \leq x \leq 5$



e) $4 \geq -x > 3$

$\therefore -4 \leq x < -3$

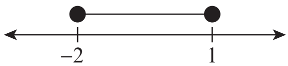


f) $16 < -2x \leq 8$

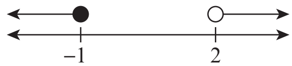
$\therefore -8 > x \geq -4$

$\therefore x < -8$ or $x \geq -4$

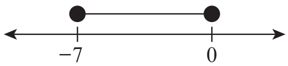
g) $-5 \leq 2 - 7x \leq 16$
$\therefore -7 \leq -7x \leq 14$
$\therefore 1 \geq x \geq -2$
$\therefore -2 \leq x \leq 1$



h) $-12 > 6 - 9x \geq 15$
$\therefore -18 > -9x \geq 9$
$\therefore 2 < x \leq -1$
$\therefore x \leq -1$ or $x > 2$

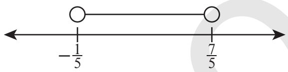


2. a) $-4 \leq 2(x + 5) \leq 10$
$\therefore -4 \leq 2x + 10 \leq 10$
$\therefore -14 \leq 2x \leq 0$
$\therefore -7 \leq x \leq 0$

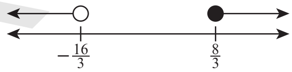


b) $1 < 3(x - 2) \leq 13$
$\therefore 1 < 3x - 6 \leq 13$
$\therefore 7 < 3x \leq 19$
$\therefore \frac{7}{3} < x \leq \frac{19}{3}$

c) $-2 < 5(1 - x) < 6$
$\therefore -2 < 5 - 5x < 6$
$\therefore -7 < -5x < 1$
$\therefore \frac{7}{5} > x > -\frac{1}{5}$
$\therefore -\frac{1}{5} < x < \frac{7}{5}$

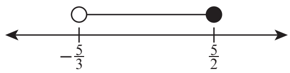


d) $-1 \leq -\frac{3x}{8} < 2$
$\therefore -8 \leq -3x < 16$
$\therefore \frac{8}{3} \leq x < -\frac{16}{3}$
$\therefore x < -\frac{16}{3}$ or $x \geq \frac{8}{3}$

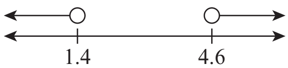


e) $-6 \leq \frac{2x}{5} - 4 < 4$
$\therefore -2 \leq \frac{2x}{5} < 8$
$\therefore -10 \leq 2x < 40$
$\therefore -5 \leq x < 10$

f) $\frac{1}{3} > -\frac{x}{5} \geq -\frac{1}{2}$
$\therefore -\frac{5}{3} < x \leq \frac{5}{2}$



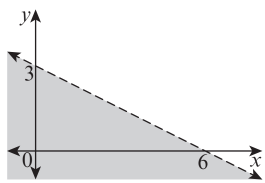
g) $-7.1 > -x - 2.5 > -3.9$
$\therefore -4.6 > -x > -1.4$
$\therefore 4.6 < x < 1.4$
$\therefore x < 1.4$ or $x > 4.6$



h) $x + 5 < 3x + 12 \leq x - 3$
$\therefore x - 7 < 3x \leq x - 15$
$\therefore -7 < 2x \leq -15$
$\therefore -\frac{7}{2} < x \leq -\frac{15}{2}$
$\therefore x \leq -\frac{15}{2}$ or $x > -\frac{7}{2}$

Exercise 8.3

1. a) y intercept: $6y = 18$
$\therefore y = 3$
x intercept: $3x = 18$
$\therefore x = 6$



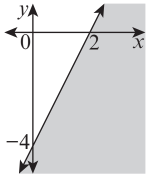
Inequality is of the form $y < \dots$, so draw dotted line and shade the region below it.

b) y intercept: $2y = -8$

$\therefore y = -4$

x intercept: $-4x = -8$

$\therefore x = 2$

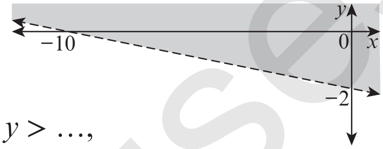


Inequality is of the form $y \leq \dots$, so draw solid line and shade the region below it.

c) y intercept: $5y = -10$

$\therefore y = -2$

x intercept: $x = -10$

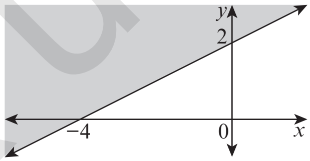


Inequality is of the form $y > \dots$, so draw dotted line and shade the region above it.

d) y intercept: $2y = 4$

$\therefore y = 2$

x intercept: $x = -4$



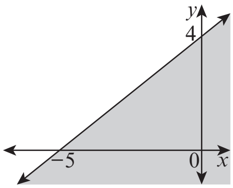
Inequality is of the form $y \geq \dots$, so draw solid line and shade the region above it.

e) y intercept: $5y = 20$

$\therefore y = 4$

x intercept: $-4x = 20$

$\therefore x = -5$



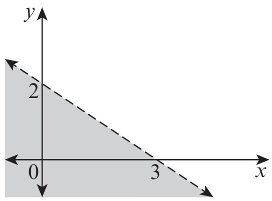
Inequality is of the form $y \leq \dots$, so draw solid line and shade the region below it.

f) y intercept: $3y = 6$

$\therefore y = 2$

x intercept: $2x = 6$

$\therefore x = 3$

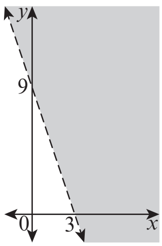


Inequality is of the form $y < \dots$, so draw dotted line and shade the region below it.

g) y intercept: $y = 9$

x intercept: $3x = 9$

$\therefore x = 3$



Inequality is of the form $y > \dots$, so draw dotted line and shade the region above it.

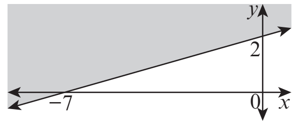
h) y intercept: $7y = 14$

$\therefore y = 2$

x intercept: $-2x = 14$

$\therefore x = -7$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.



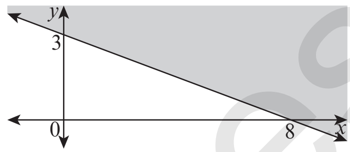
i) y intercept: $8y = 24$

$\therefore y = 3$

x intercept: $3x = 24$

$\therefore x = 8$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.



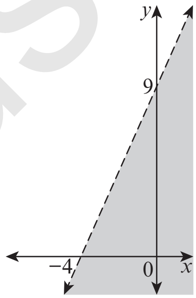
j) y intercept: $4y = 36$

$\therefore y = 9$

x intercept: $-9x = 36$

$\therefore x = -4$

Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.

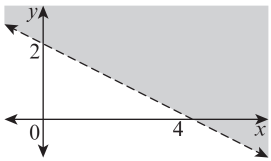


2. **a)** y intercept: $y = 2$

x intercept: $0.5x = 2$

$\therefore x = 4$

Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.



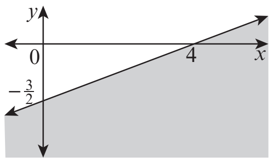
b) y intercept: $\frac{2}{3}y = -1$

$\therefore y = -\frac{3}{2}$

x intercept: $-\frac{1}{4}x = -1$

$\therefore x = 4$

Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.



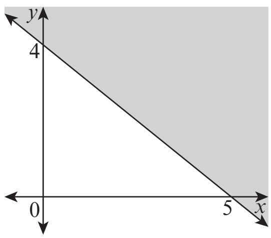
c) y intercept: $\frac{5}{6}y = \frac{10}{3}$

$\therefore y = 4$

x intercept: $\frac{2}{3}x = \frac{10}{3}$

$\therefore x = 5$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.



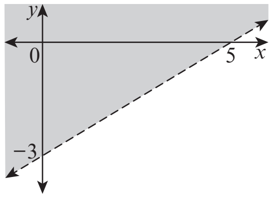
d) y intercept: $1.25y = -3.75$

$\therefore y = -3$

x intercept: $0.75x = 3.75$

$\therefore x = 5$

Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.



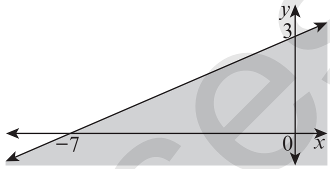
e) y intercept: $0.7y = 2.1$

$\therefore y = 3$

x intercept: $-0.3x = 2.1$

$\therefore x = -7$

Inequality is of the form $y \le \ldots$, so draw solid line and shade the region below it.



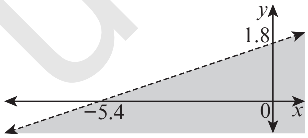
f) y intercept: $0.375y = 0.675$

$\therefore y = 1.8$

x intercept: $0.125x = -0.675$

$\therefore x = -5.4$

Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.



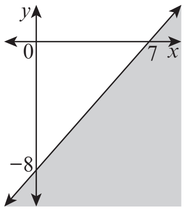
g) y intercept: $-\frac{1}{8}y = 1$

$\therefore y = -8$

x intercept: $\frac{1}{7}x = 1$

$\therefore x = 7$

Inequality is of the form $y \le \ldots$, so draw solid line and shade the region below it.



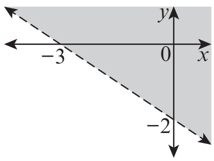
h) y intercept: $0.3y = -0.6$

$\therefore y = -2$

x intercept: $0.2x = -0.6$

$\therefore x = -3$

Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.



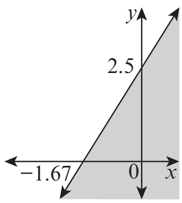
i) y intercept: $1.4y = 3.5$

$\therefore y = 2.5$

x intercept: $-2.1x = 3.5$

$\therefore x = -1.67$

Inequality is of the form $y \le \ldots$, so draw solid line and shade the region below it.

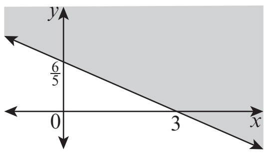


j) y intercept: $\frac{5}{8}y = \frac{3}{4}$

$\therefore y = \frac{6}{5}$

x intercept: $\frac{1}{4}x = \frac{3}{4}$

$\therefore x = 3$



Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

Exercise 8.4

1. $-5x + 3y \leq 15$:

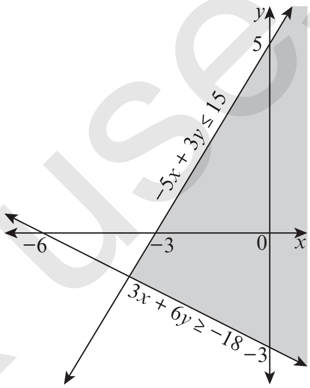
y intercept: $3y = 15$

$\therefore y = 5$

x intercept: $-5x = 15$

$\therefore x = -3$

Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.



$3x + 6y \geq -18$:

y intercept: $6y = -18$

$\therefore y = -3$

x intercept: $3x = -18$

$\therefore x = -6$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

2. $7x + 2y < 14$:

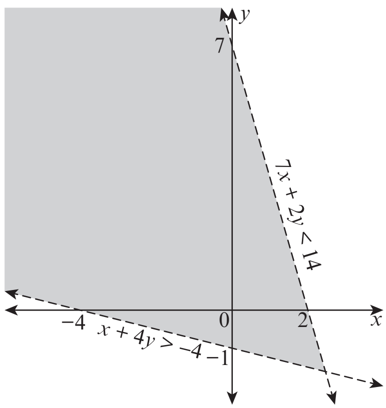
y intercept: $2y = 14$

$\therefore y = 7$

x intercept: $7x = 14$

$\therefore x = 2$

Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.



$x + 4y > -4$:

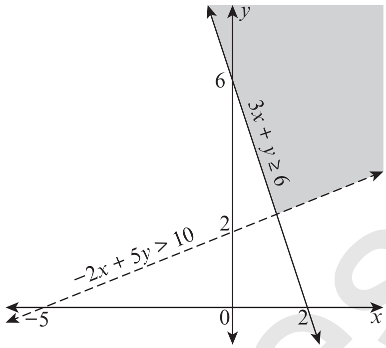
y intercept: $4y = -4$

$\therefore y = -1$

x intercept: $x = -4$

Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.

3. $-2x + 5y > 10$:



y intercept: $5y = 10$

$\therefore y = 2$

x intercept: $-2x = 10$

$\therefore x = -5$

Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.

$3x + y \geq 6$:

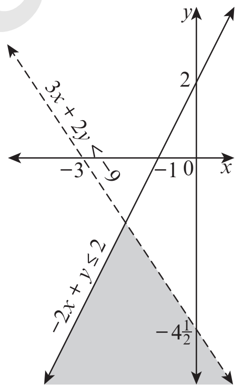
y intercept: $y = 6$

x intercept: $3x = 6$

$\therefore x = 2$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

4. $-2x + y \leq 2$:



y intercept: $y = 2$

x intercept: $-2x = 2$

$\therefore x = -1$

Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.

$3x + 2y < -9$:

y intercept: $2y = -9$

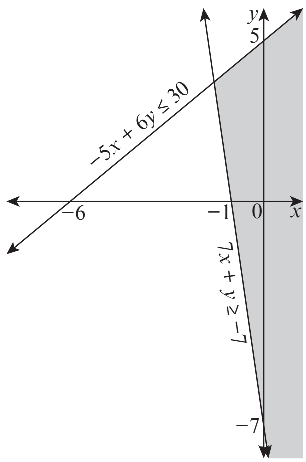
$\therefore y = -\frac{9}{2}$

x intercept: $3x = -9$

$\therefore x = -3$

Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.

5. $-5x + 6y \leq 30$:



y intercept: $6y = 30$

$\therefore y = 5$

x intercept: $-5x = 30$

$\therefore x = -6$

Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.

$7x + y \geq -7$:

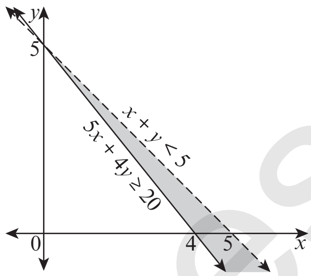
y intercept: $y = -7$

x intercept: $7x = -7$

$\therefore x = -1$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

6. $5x + 4y \geq 20$:



y intercept: $4y = 20$

$\therefore y = 5$

x intercept: $5x = 20$

$\therefore x = 4$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

$x + y < 5$:

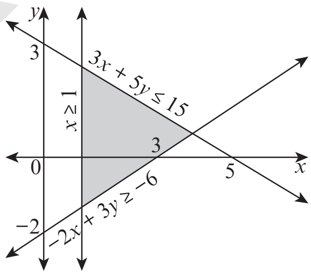
y intercept: $y = 5$

x intercept: $x = 5$

Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.

Exercise 8.5

1. $3x + 5y \leq 15$:



y intercept: $5y = 15$

$\therefore y = 3$

x intercept: $3x = 15$

$\therefore x = 5$

Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.

$-2x + 3y \geq -6$:

y intercept: $3y = -6$

$\therefore y = -2$

x intercept: $-2x = -6$

$\therefore x = 3$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

$x \geq 1$: Draw solid line and shade the region to the right of it.

2. $6x + 5y < 30$:

y intercept: $5y = 30$

$\therefore y = 6$

x intercept: $6x = 30$

$\therefore x = 5$

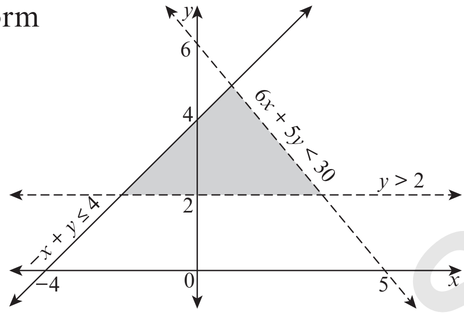
Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.

$-x + y \leq 4$:

y intercept: $y = 4$

x intercept: $-x = 4$

$\therefore x = -4$



Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.

$y > 2$: Draw dotted line and shade the region above it.

3. $x + y \leq 3$:

y intercept: $y = 3$

x intercept: $x = 3$

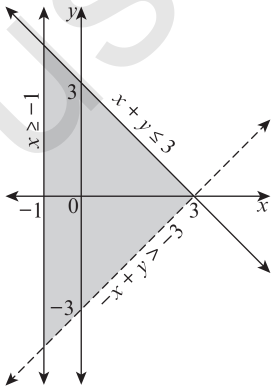
Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.

$-x + y > -3$:

y intercept: $y = -3$

x intercept: $-x = -3$

$\therefore x = 3$



Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.

$x \geq -1$: Draw solid line and shade the region to the right of it.

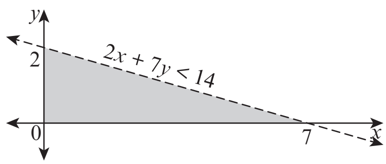
4. $2x + 7y < 14$:

y intercept: $7y = 14$

$\therefore y = 2$

x intercept: $2x = 14$

$\therefore x = 7$



Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.

$x \geq 0$: Shade the region to the right of the y axis.

$y \geq 0$: Shade the region above the x axis.

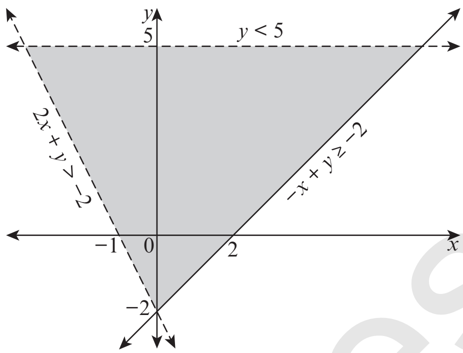
5. $-x + y \geq -2$:

y intercept: $y = -2$

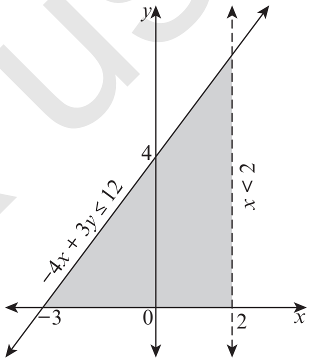
x intercept: $-x = -2$

$\therefore x = 2$

Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.
$2x + y > -2$:
y intercept: $y = -2$
x intercept: $2x = -2$
$\therefore x = -1$
Inequality is of the form $y > \ldots$, so draw dotted line and shade the region above it.
$y < 5$: Draw dotted line and shade the region below it.



6. $-4x + 3y \leq 12$:
y intercept: $3y = 12$
$\therefore y = 4$
x intercept: $-4x = 12$
$\therefore x = -3$
Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.
$y \geq 0$: Shade the region above the x axis.
$x < 2$: Draw this line and shade the region to the left of it.



Exercise 8.6

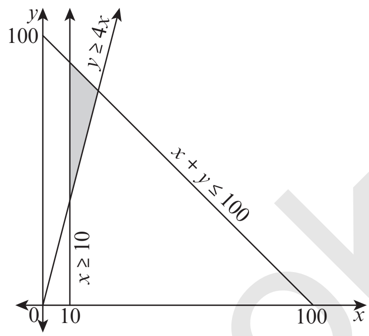
1. The factory cannot make a negative number of tables or chairs, so $x \geq 0$ and $y \geq 0$.
The factory can make at most 100 items each week, so $x + y \leq 100$.
The factory must make at least 10 tables every week, so $x \geq 10$. (Note that this new constraint means that the constraint $x \geq 0$ falls away.)
For each table, there must be at least four chairs, so $y \geq 4x$.
$\therefore$ The constraints are: $y \geq 0$, $x + y \leq 100$, $x \geq 10$ and $y \geq 4x$.

2. The farmer cannot plant a negative number of hectares, so $x \geq 0$ and $y \geq 0$.
She has at most 50 hectares of farmland on which she can plant crops, so $x + y \leq 50$.

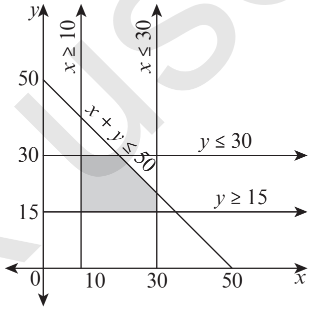
He must plant at least 10 hectares under cassava, so $x \geq 10$.
She must plant at least 15 hectares under yams, so $y \geq 15$.
(Notice that these new constraints mean that the first two constraints fall away.)
She must plant at most 30 hectares under cassava, so $x \leq 30$.
She must plant at most 30 hectares under yams, so $y \leq 30$.
$\therefore$ The constraints are: $x + y \leq 50$, $x \geq 10$, $y \geq 15$, $x \leq 30$ and $y \leq 30$.

Exercise 8.7

1. a) and **b)**



2. a) and **b)**



Exercise 8.8

1. A(10; 40), B(10; 90), C(20; 80)

2. A(10; 15), B(10; 30), C(20; 30), D(30; 20), E(30; 30)

Exercise 8.9

1. a) $M = 9x + 7y$ **b)** $C = 140\ 000x + 95\ 000y$
c) $P = 65\ 000x + 42\ 000y$

2. a) $L = 5x + 30y$ **b)** $T = 18x + 30y$

Exercise 8.10

1. a) At vertex A(0; 0): C = 98 000(0) + 69 000(0) = ₦0
At vertex B(0; 160): C = 98 000(0) + 69 000(160)
= ₦11 040 000
At vertex C(50; 125): C = 98 000(50) + 69 000(125)
= ₦13 525 000
$\therefore$ The cost is a maximum at vertex C and a minimum at vertex A.

b) At vertex A(0; 0): M = 40(0) + 60(0)
= 0 working hours

At vertex B(0; 160): M = 40(0) + 60(160)
= 9 600 working hours
At vertex C(50; 125): M = 40(50) + 60(125)
= 9 500 working hours
So the working hours are a maximum at vertex B and a minimum at vertex A.

c) At vertex A(0; 0): A = 5(0) + 2.5(0) = 0 m^2
At vertex B(0; 160): A = 5(0) + 2.5(160) = 400 m^2
At vertex C(50; 125): A = 5(50) + 2.5(125) = 562.5 m^2
∴ The storage space is a maximum at vertex C and a minimum at vertex A.

2. a) At vertex A(15; 25): F = 800(15) + 750(25) = 30 750 ℓ
At vertex B(15; 195): F = 800(15) + 750(195)
= 158 250 ℓ
At vertex C(30; 170): F = 800(30) + 750(170)
= 151 500 ℓ
At vertex D(45; 115): F = 800(45) + 750(115)
= 122 250 ℓ
∴ The fuel is a maximum at vertex B and a minimum at vertex A.

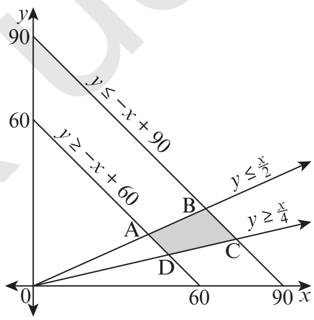
b) At vertex A(15; 25): T = 20(15) + 35(25)
= 1 175 hours
At vertex B(15; 195): T = 20(15) + 35(195)
= 7 125 hours
At vertex C(30; 170): T = 20(30) + 35(170)
= 6 550 hours
At vertex D(45; 115): T = 20(45) + 35(115)
= 4 925 hours
∴ The maintenance time is a maximum at vertex B and a minimum at vertex A.

c) At vertex A(15; 25): M = 10(15) + 5(25) = 275 tonnes
At vertex B(15; 195): M = 10(15) + 5(195)
= 1 125 tonnes
At vertex C(30; 170): M = 10(30) + 5(170)
= 1 150 tonnes
At vertex D(45; 115): M = 10(45) + 5(115)
= 1 025 tonnes
∴ The mass is a maximum at vertex C and a minimum at vertex A.

Exercise 8.11

1. a) It is not possible to use a negative number of pieces of fruit, so $x \geq 0$ and $y \geq 0$.
Every batch must contain at least 60 pieces of fruit, so $x + y \geq 60$.
Every batch must contain at most 90 pieces of fruit, so $x + y \leq 90$.
The number of mangoes used must be at least twice the number of guavas used, so $x \geq 2y$.
The number of mangoes used must not be more than four times the number of guavas used, so $x \leq 4y$.
$\therefore$ The constraints are: $x \geq 0$, $y \geq 0$, $x + y \geq 60$, $x + y \leq 90$, $x \geq 2y$ and $x \leq 4y$.

b) If we make y the subject of the last four inequalities, we get: $y \geq -x + 60$, $y \leq -x + 90$, $y \leq \frac{x}{2}$ and $y \geq \frac{x}{4}$.

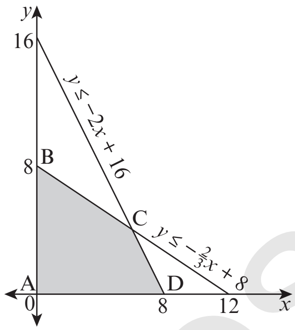


c) A(40; 20), B(60; 30), C(72; 18), D(48; 12)

d) Objective function:
C = $250x + 150y$,
where C is the cost in naira
At vertex A(40; 20): C = 250(40) + 150(20) = ₦13 000
At vertex B(60; 30): C = 250(60) + 150(30) = ₦19 500
At vertex C(72; 18): C = 250(72) + 150(18) = ₦20 700
At vertex D(48; 12): C = 250(48) + 150(12) = ₦13 800
$\therefore$ The cost is a minimum at vertex A, so 40 mangoes and 20 guavas should be used.

2. a) It is not possible to have a negative number of assembly lines, so $x \geq 0$ and $y \geq 0$.
Type A occupies 40 m^2 of floor space and Type B occupies 60 m^2 of floor space. There is a total amount of 480 m^2 of floor space available, so $40x + 60y \leq 480$.
Type A needs 8 skilled workers and Type B needs 4 skilled workers. There are 64 skilled workers available, so $8x + 4y \leq 64$.
$\therefore$ The constraints are: $x \geq 0$, $y \geq 0$, $40x + 60y \leq 480$ and $8x + 4y \leq 64$.

b) If we make y the subject of the last two inequalities, we get:
$y \leq -\frac{2}{3}x + 8$ and $y \leq -2x + 16$.



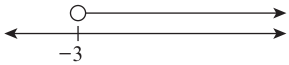
c) A(0; 0), B(0; 8), C(6; 4), D(8; 0)

d) Objective function:
N = $5x + 3y$, where N is the number of bicycles produced daily
At vertex A(0; 0): N = 5(0) + 3(0) = 0
At vertex B(0; 8): N = 5(0) + 3(8) = 24
At vertex C(6; 4): N = 5(6) + 3(4) = 42
At vertex D(8; 0): N = 5(8) + 3(0) = 40
$\therefore$ The number of bicycles is a maximum at vertex C, so 6 of Type A and 4 of Type B should be installed.

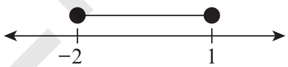
Assess your progress

1. **a)** $6x - 5 \geq 19$
$\therefore 6x \geq 24$
$\therefore x \geq 4$

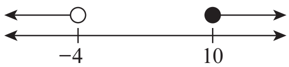
b) $x - 5 < 4(x + 1)$
$\therefore x - 5 < 4x + 4$
$\therefore -3x < 9$
$\therefore x > -3$



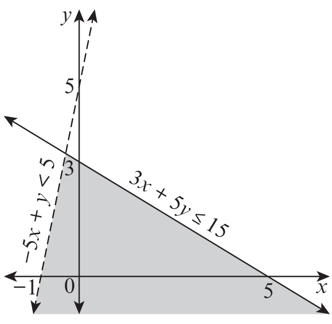
c) $-1 \leq 2x + 3 \leq 5$
$\therefore -4 \leq 2x \leq 2$
$\therefore -2 \leq x \leq 1$



d) $0 < -\frac{x}{2} - 2 \leq -7$
$\therefore 2 < -\frac{x}{2} \leq -5$
$\therefore 4 < -x \leq -10$
$\therefore -4 > x \geq 10$
$\therefore x < -4$ or $> x \geq 10$

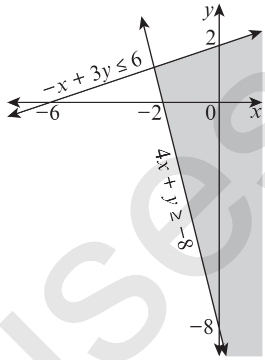


2. **a)** $-5x + y < 5$:
y intercept: $y = 5$
x intercept: $-5x = 5$
$\therefore x = -1$
Inequality is of the form $y < \ldots$, so draw dotted line and shade the region below it.
$3x + 5y \leq 15$:
y intercept: $5y = 15$
$\therefore y = 3$



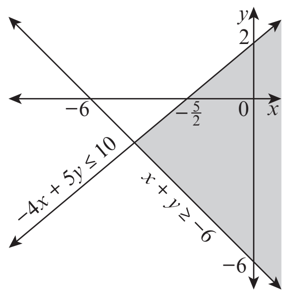
x intercept: $3x = 15$
$\therefore x = 5$
Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.

b) $-x + 3y \leq 6$:
y intercept: $3y = 6$
$\therefore y = 2$
x intercept: $-x = 6$
$\therefore x = -6$
Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.



$4x + y \geq -8$:
y intercept: $y = -8$
x intercept: $4x = -8$
$\therefore x = -2$
Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

c) $-4x + 5y \leq 10$:
y intercept: $5y = 10$
$\therefore y = 2$
x intercept: $-4x = 10$
$\therefore x = -\frac{5}{2}$
Inequality is of the form $y \leq \ldots$, so draw solid line and shade the region below it.



$x + y \geq -6$:
y intercept: $y = -6$
x intercept: $x = -6$
Inequality is of the form $y \geq \ldots$, so draw solid line and shade the region above it.

3. $C = 350x + 400y$

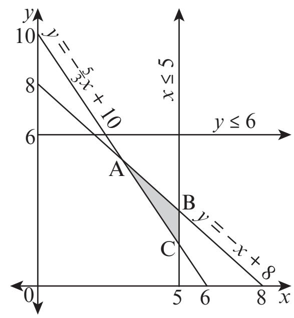
4. a) At vertex A(0; 15): C = 6 000(0) + 5 550(15) = ₦83 250
At vertex B(5; 155): C = 6 000(5) + 5 550(155)
= ₦890 250
At vertex C(250; 30): C = 6 000(250) + 5 550(30)
= ₦1 666 550
At vertex D(75; 0): C = 6 000(75) + 5 550(0)
= ₦450 000
$\therefore$ The cost is a maximum at vertex C and a minimum at vertex A.

b) At vertex A(0; 15): P = 1 250(0) + 950(15) = ₦14 250
At vertex B(5; 155): P = 1 250(5) + 950(155)
= ₦153 500
At vertex C(250; 30): P = 1 250(250) + 950(30)
= ₦341 000
At vertex D(75; 0): P = 1 250(75) + 950(0) = ₦93 750
∴ The profit is a maximum at vertex C and a minimum at vertex A.

c) At vertex A(0; 15): M = 15(0) + 12(15)
= 180 working hours
At vertex B(5; 155): M = 15(5) + 12(155)
= 1 935 working hours
At vertex C(250; 30): M = 15(250) + 12(30)
= 4 110 working hours
At vertex D(75; 0): M = 15(75) + 12(0)
= 1 125 working hours
∴ The working hours are a maximum at vertex C and a minimum at vertex A.

5. a) It is not possible to rent a negative number of buses, so $x \geq 0$ and $y \geq 0$.
Type A has seating for 50 students and Type B has seating for 30 students.
300 students must be transported, so
$50x + 30y \geq 300$.
There are 5 Type A buses, so $x \leq 5$.
There are 6 Type B buses, so $y \leq 6$.
8 people are available to drive, ∴ $x + y \leq 8$.

b)



c) The constraint $y \leq 6$ does not play a part in the solution, because it has no effect on the feasible region.

d) A(3; 5), B(5; 3), C(5; 1.7)

e) $C = 2\ 500x + 1\ 200y$

f) At vertex A(3; 5): C = 2 500(3) + 1 200(5) = ₦13 500
At vertex B(5; 3): C = 2 500(5) + 1 200(3) = ₦16 100
It is not possible to rent a fraction of a bus, so the coordinates of C become (5; 2).
At vertex C(5; 2): C = 2 500(5) + 1 200(2) = ₦14 900
The cost is a minimum at vertex A, so the school should rent 3 of Type A and 5 of Type B.

TOPIC 9

Algebraic fractions

Introduction

Begin by simplifying algebraic fractions using the addition, subtraction, multiplication and division operations. Explain that we also use factorisation to simplify algebraic fractions.

The next section deals with solving equations containing fractions, and uses these in real-life problem solving, for example travel problems involving distance, speed and time, as well as work problems.

Common difficulties

Students need to recognise when to factorise an expression and remember the different methods of factorisation.

Some students battle to find a common denominator in an expression or equation so will need to revise that section of work.

The order of operations is very important when solving equations involving fractions, so make sure students bear this in mind.

Remind students that they should first get rid of any fractions by multiplying by the LCM of the denominators and then solve the equation in the usual way.

Make sure students understand why dividing by a fraction is the same as multiplying by the inverse. Use an example such as 1 divided by $\frac{1}{2}$, which is the same as asking 'How many halves are there in 1?'.

$1 \div \frac{1}{2}$ is equal to $1 \times \frac{2}{1}$, which means there are two halves in 1.

Preparation

Prepare a chart that explains the vocabulary used in this topic:

- A **common fraction** has a numerator and a denominator.
- In a **proper fraction**, the numerator is less than the denominator, for example $\frac{2}{5}$.
- In an **improper fraction** the numerator is greater than the denominator, for example $\frac{9}{4}$.

- A **mixed number** or **mixed fraction** has a number part and a fraction part, for example $3\frac{1}{4}$.
- An **algebraic fraction** is any of the types of fractions described above, but it contains a **variable**.

You could also provide a chart showing the different methods of factorisation of algebraic fractions, and a chart of the graph of a hyperbola, which will illustrate the fact that we cannot divide by zero.

Introduction for students

Revise basic numerical fractions with a few examples of each different type of fraction. Make sure students are able to simplify fractions.

Get students to write equivalent fractions for fractions of different types.

Explain that the basic rules for working with fractions still apply when the fractions contain variables.

Work through the different methods of simplifying fractions. Go through the examples illustrating the use of mathematical operations on fractions and the rules to remember.

Answers

Exercise 9.1

1. **a)** $x = 13$ **b)** $x = 5$ **c)** $x = 24$
d) $x = 120$ **e)** $x = 12$ **f)** $x = 60$

2. **a)** $\frac{2}{3}$ **b)** $\frac{1}{3}$ **c)** $\frac{2}{5}$ **d)** $\frac{7}{8}$
e) $\frac{5}{3}$ **f)** $\frac{23}{4}$ **g)** $\frac{15}{4}$ **h)** $\frac{2}{15}$

3. **a)** $\frac{5}{6} = \frac{-15}{\mathbf{-18}} = \frac{\mathbf{300}}{360} = \frac{25}{\mathbf{30}}$ **b)** $\frac{2}{30} = \frac{60}{\mathbf{900}} = \frac{-10}{\mathbf{-150}} = \frac{\mathbf{5}}{75}$
c) $\frac{3}{24} = \frac{-12}{\mathbf{-96}} = \frac{\mathbf{5}}{40} = \frac{24}{\mathbf{192}}$ **d)** $\frac{70}{\mathbf{280}} = \frac{25}{\mathbf{100}} = \frac{14}{56} = \frac{\mathbf{12}}{48}$

4. **a)** $\frac{23}{12}$ **b)** $\frac{1}{2}$ **c)** $\frac{7}{20}$ **d)** $\frac{49}{24}$
e) $\frac{1}{5}$ **f)** undefined **g)** $\frac{9}{5}$ **h)** $\frac{1}{5}$
i) 6 **j)** $\frac{1}{2}$

Exercise 9.2

1. **a)** $\frac{2}{3}$ **b)** 3 **c)** $\frac{x}{3}$ **d)** $\frac{1}{6}$
e) 6 **f)** $1 - 2a$ **g)** b^3 **h)** $\frac{4}{c^2}$
i) 0 **j)** undefined

2. **a)** $\frac{1}{2}$ **b)** -1 **c)** y **d)** $\frac{x+y}{x-y}$
e) $\frac{x-1}{x+1}$ **f)** $\frac{x-1}{2x-1}$ **g)** $2x + 3y$ **h)** $18x^2y$

3. **a)** $2x + 3$ **b)** $2x^2 - 4$
c) $\frac{x-3}{x+2}$ **d)** $\frac{x+3}{x-4}$
e) $\frac{(x-2-1)(x-2+1)}{(x-3)(x-1)} = 1$ **f)** $\frac{x-2}{x-3}$

Exercise 9.3

1. **a)** $\frac{7x}{4}$

b) $\frac{3x-3-2x-2}{6} = \frac{x-5}{6}$

c) $\frac{4x+6+5x-5+10}{10} = \frac{9x+11}{10}$

d) $\frac{20x-12-5x+20}{20} = \frac{15x+8}{20}$

e) $\frac{6x+8x+x+2}{4} = \frac{15x+2}{4}$

f) $\frac{6x+3-2x+12}{6} = \frac{4x+15}{6}$

g) $\frac{2x+15x-30x+10}{10} = \frac{-13x+10}{10}$

h) $\frac{4-2x-9x+12}{6} = \frac{-11x+16}{6}$

i) $\frac{2x-4+3x-9}{12} = \frac{5x-13}{12}$

j) $\frac{8}{x}$

2. **a)** $\frac{2b+3a}{ab}$

b) $\frac{2x+2-4x+12}{(x-3)(x+1)} = \frac{-2x+14}{(x-3)(x+1)}$

c) $\frac{3x-3+2x-8}{(x-4)(x-1)} = \frac{5x-11}{(x-4)(x-1)}$

d) $\frac{2+2y-2y+1}{2} = \frac{3}{2}$

e) $\frac{-6x-3+x-2}{(x-2)(2x+1)} = \frac{-5x-5}{(x-2)(2x+1)}$

f) $\frac{1}{(x-1)(x+1)} + \frac{1}{x(x-1)} = \frac{x+x+1}{x(x-1)(x+1)} = \frac{2x+1}{(x(x-1)(x+1)}$

g) $\frac{1}{(x-5)(x+5)} - \frac{2}{x+5} = \frac{1-2x+10}{(x-5)(x+5)} = \frac{-2x+11}{(x-5)(x+5)}$

h) $\frac{2}{1-x} + \frac{2x+4}{(x+2)(x-1)} = \frac{-2x-4+2x+4}{(x+2)(x-1)} = 0$

i) $\frac{x+1+x+2-x}{x(x+1)(x+2)} = \frac{x+3}{x(x+1)(x+2)}$

j) $\frac{-(a-1)^2+(a-2)^2+3a-5}{(a-2)(a-1)} = \frac{a-2}{(a-2)(a-1)} = \frac{1}{a-1}$

Exercise 9.4

1. **a)** $\frac{3}{4y}$ **b)** $\frac{2a^3}{25}$ **c)** $\frac{4}{3}$ **d)** $\frac{2}{a}$

e) $3xy - 6y$ **f)** 2 **g)** 3 **h)** 1

2. **a)** $\frac{2(y-1)}{y(y-6)} \times \frac{(y-6)(y+1)}{(y-1)(y+1)} = \frac{2}{y}$

b) $\frac{4a(a+2)}{(a-2)(a+2)} \times \frac{(a-3)(a-2)}{4(a-3)} = a$

c) $\frac{(x-4)(x+4)}{x(x-4)} \times \frac{3(x-1)}{(x+4)(x-1)} = \frac{3}{x}$

d) $\frac{5y(x-3)}{4(x-3)} \times \frac{x+y}{6y^2} = \frac{5(x+y)}{24y}$

e) $\frac{x(x+1)}{(x-3)(x+1)} \times \frac{(x-3)(x+1)}{-(x-1)(x+1)} = -\frac{x}{x+1}$

f) $\frac{(x-2)(x+1)}{(x-2)(x+2)} \times \frac{x(x+2)}{x(x+1)} = 1$

g) $\frac{(a-1)(a+1)}{(2a-3)(a-1)} \times \frac{3(2a-3)}{4(a+1)}] \times \frac{4a}{3} = a$

h) $\frac{(2a+1)(a-1)}{(2a-3)(2a+1)} \times \frac{(3a-2)(2a-3)}{(2a-3)(a-1)} = \frac{3a-2}{2a-3}$

3. **a)** $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} = \frac{\frac{y+x}{xy}}{\frac{(x-y)(x+y)}{xy}} = \frac{1}{x-y}$ **b)** $\frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{\frac{x+1}{x^2}}{\frac{(x-1)(x+1)}{x^2}} = \frac{1}{x-1}$

Exercise 9.5

1. 13 **2.** $\frac{1}{14}$ **3.** $\frac{3b-8}{8b-5}$ **4.** a

5. $\frac{4}{3}$ **6.** 1 **7.** $\frac{1}{2}$ **8.** $\frac{5}{2}$

9. $\frac{2a}{1-a}$ **10.** -7

Exercise 9.6

1. **a)** $x = 4$ **b)** $x = 7$ or $x = -14$

2. **a)** $x = \frac{1}{2}$ **b)** $x = 3$ or $x = 5$

c) $x = \frac{-1}{3}$ or $x = \frac{1}{2}$

3. **a)** $x = -1$ or $x = -\frac{3}{2}$ **b)** $x = \frac{7}{3}$ or $x = 1$

4. 50 km/h

5. 60 km/h

6. The apprentice takes 60 days and the painter takes 40 days.

Exercise 9.7

1. $x = -3$
2. $x = \frac{1}{2}$
3. $x = 5$
4. $x = 2$ or $x = 0$
5. $x = 7$ or $x = -\frac{1}{2}$
6. $x = -6$ or $x = \frac{1}{3}$
7. $x = \pm 1$ or $x = -\frac{3}{4}$
8. $x = 1$; 2 or -7
9. $x = -2$; -3; -5 or 10
10. $x = \pm 2$; 1 or $\frac{1}{2}$

Assess your progress

1. **a)** $a = 2$ **b)** $b = 18$ **c)** $c = 3$ **d)** $d = 10$

2. **a)** $\frac{1}{6}$ **b)** $2x$ **c)** x **d)** $\frac{x-y}{x+y}$

3. **a)** $\frac{21x}{6} = \frac{7x}{2}$ **b)** $\frac{8x+7}{6}$

   c) $\frac{(x-y)(x+y)}{3(x-y)} \times \frac{1}{x+y} = \frac{1}{3}$ **d)** $\frac{2xy(2x+y)}{3(2x+y)} \times \frac{3}{xy} = 2$

4. $\frac{\frac{5}{4}}{\frac{5}{10}} = \frac{5}{2}$

5. **a)** $\frac{-2(x-4) - 16 + x(x+4)}{(x+4)(x-4)}$

   $= \frac{x^2 + 2x - 8}{(x+4)(x-4)}$

   $= \frac{(x-2)(x+4)}{(x+4)(x-4)}$

   $= \frac{(x-2)}{(x+4)}$

   b) $\frac{2(y-1)}{y(y-6)} \times \frac{(y-6)(y+1)}{(y+1)(y-1)} = \frac{2}{y}$

   c) $\frac{4(a+3)}{(2a-1)(a+3)} \times \frac{2a-1}{2} = 2$

   d) $\frac{4(2x-3y)}{2(x-y)} \times \frac{(x-y)(x+y)}{(2x-3y)(x+y)}$
   $= 2$

6. $\frac{x^2 + 2x + 1}{x^3} = \frac{(x+1)^2}{x^3}$

7. **a)** $x = 3$ **b)** $x = \frac{6}{5}$
   c) $x = 1$ or $x = -1$

8. **a)** $x = 4$ or $x = -\frac{3}{2}$ **b)** -5; $\frac{5}{4}$
   c) $-\frac{1}{7}$

9. $x = \frac{3}{2}$

10. $\frac{x}{y} = \frac{1}{2}$ or $\frac{x}{y} = -6$

TOPIC 10 Logical reasoning

Introduction

In this topic your students will revisit logical reasoning, having been introduced to it in SS1. Begin by revising the basic components of logical reasoning.

In the next section students will define the converse, inverse and contrapositive of a conditional statement. They will also learn about equivalent statements.

Your students will then apply contrapositives and inverses in proving theories.

Common difficulties in this topic

One of the chief difficulties in understanding logical reasoning is that at this basic level, many examples can feel contrived. Also students may find it difficult to distinguish between converses, inverses and contrapositives.

Work through as many examples as possible with your class, including some of your own, where necessary.

If some of your students struggle with the abstract concepts involved, allow them to partner with a student who is stronger in this respect. In this way, students learn to work together and to help one another.

Some students will struggle to interpret the written instructions for the more complex constructions. Be prepared to remediate this problem by demonstrating these steps to those learners who are struggling.

Students sometimes interpret a question incorrectly and consequently do not answer what is being asked. Encourage your students to read through every question at least twice to be sure about what is being asked of them before they decide how to proceed.

Preparation

Prepare the following charts:

- examples of simple statements, true and false statements and negation of statements

- truth tables for the five logical operations: negation, conjunction, disjunction, implication and bi-implication
- examples of compound statements involving conjunction, disjunction, implication and bi-implication
- the logic symbols for the converse, inverse and contrapositive of a conditional statement, as well as the equivalence of two statements:
  - $\sim$ means 'not'
  - $\wedge$ means 'and'
  - $\vee$ means 'or'
  - $\Rightarrow$ means 'if …, then …'
  - $\Leftrightarrow$ means '…, if and only if …'
- an outline of the two methods used in indirect proofs.

Introduction for students

Write the logic symbols on the board: $\sim$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$. Ask your class if they can remember what each one means.

Now write a few logical statements on the board, for example:

$\sim P$	$\sim Q$	$P \wedge Q$	$P \vee Q$
$\sim\sim Q$	$P \Rightarrow Q$	$Q \Leftrightarrow P$	$P \Rightarrow \sim Q$

Ask for volunteers from your class to read each one out loud. For example, $P \Rightarrow Q$ means: 'If P, then not Q'. See how many of these your class are able to answer on their own, without help from you.

Answers

Exercise 10.1

1. **a)** This is a closed statement. It is false.
   b) This is a closed statement. It is true.
   c) This is an open statement.
   d) This is a closed statement. It is true.
   e) This is a closed statement. It is true.
   f) This is an open statement.
   g) This is a closed statement. It is false.
   h) This is a closed statement. It is true.

2. **a)** Some even integers are divisible by 3.
   b) $13 - 5 \neq 8$

c) Next Wednesday is not a public holiday.
d) No integers are negative.
e) 17 and 18 are not consecutive numbers.
f) $x \leq 45$
g) Some integers are not natural numbers.
h) All children are fond of vegetables.

Exercise 10.2

1. a) $x = 6$ and $y = 4$ or $x = 8$ and $y = 3$.
   b) $x = 4$ and $y = 6$ or any other combination such that either P or Q or both are false.
   c) $x = 5$ and $y = 2$ or any other combination such that either P or Q or both are true.
   d) $x = 1$ and $y = 2$ or any other combination such that both P and Q are false.

2. a) It is not raining.
   b) The roof is not leaking.
   c) It is raining and the roof is leaking.
   d) It is raining or the roof is leaking.
   e) The roof is leaking.
   f) If it is raining, then the roof is leaking.
   g) The roof is leaking if and only if it is raining.
   h) If it is raining, then the roof is not leaking.

3.

P	Q	P ∨ Q	~Q	~Q ⇒ P	(P ∨ Q) ⇒ (~Q ⇒ P)
T	T	T	F	T	T
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	F	T

All possible truth values for this conditional statement are true, so the conditional statement is a tautology.

4. a)

P	Q	~P	~P ∨ Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

b)

P	Q	~P	~Q	~P ∧ ~Q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

c)

P	Q	P ⇒ Q	P ∨ (P ⇒ Q)
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

d)

P	Q	~P	~P ∧ Q	~Q	(~P ∧ Q) ∨ ~Q
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

e)

P	Q	P ∧ Q	(P ∧ Q) ⇒ P
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

f)

P	Q	~P	~Q	~P ∨ ~Q	P ∧ Q	(~P ∨ ~Q) ∧ (P ∧ Q)
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

5. The compound statements in Questions 4 c) and e) are tautologies, because all their possible truth values are true.

Exercise 10.3

1. a) Let P be the statement ‘A shape is a square’ and Q be the statement ‘The shape is a quadrilateral’.
   In logical notation: P ⇒ Q.
   This statement is true.

b) In logical notation: Q ⇒ P.
In words: If a shape is a quadrilateral, then the shape is a square.
This statement is false, because there are many examples of quadrilaterals (parallelograms, kites, trapeziums, rectangles, rhombuses and so on) that are not squares.

c) In logical notation: ~P ⇒ ~Q.
In words: If a shape is not a square, then the shape is not a quadrilateral.
This statement is false, because the shape could be any other kind of quadrilateral, for example a rectangle.

d) In logical notation: ~Q ⇒ ~P.
In words: If a shape is not a quadrilateral, then the shape is not a square.
This statement is true, because any shape that is not a quadrilateral is definitely not a square.

2. **a)** Let P be the statement 'A polygon has three sides' and Q be the statement 'The polygon is a triangle'.
In logical notation: P ⇒ Q.
This statement is true.

b) In logical notation: Q ⇒ P.
In words: If a polygon has three sides, then it is a triangle.
This statement is true.

c) In logical notation: ~P ⇒ ~Q.
In words: If a polygon does not have three sides, then it is not a triangle.
This statement is true.

d) In logical notation: ~Q ⇒ ~P.
In words: If a polygon is not a triangle, then it does not have three sides.
This statement is true.

3. **a)** Let P be the statement 'x is a factor of 6' and Q be the statement 'x is a factor of 12'.
In logical notation: P ⇒ Q.
This statement is true.

b) In logical notation: Q ⇒ P.
In words: If x is a factor of 12, then x is a factor of 6.
This statement is false, because 12 is a factor of 12, but not of 6.

c) In logical notation: ~P ⇒ ~Q.
In words: If x is not a factor of 6, then x is not a factor of 12.
This statement is false, because 12 is not a factor of 6, but it is a factor of 12.

d) In logical notation: ~Q ⇒ ~P.
In words: If x is not a factor of 12, then x is not a factor of 6.
This statement is true, because any number that is not a factor of 12 is also definitely not a factor of 6.

4. a) Let P be the statement 'x is an even prime number' and Q be the statement 'x is 2'.
In logical notation: P ⇒ Q.
This statement is true.

b) In logical notation: Q ⇒ P.
In words: If x is 2, then x is an even prime number.
This statement is true.

c) In logical notation: ~P ⇒ ~Q.
In words: If x is not an even prime number, then x is not 2.
This statement is true.

d) In logical notation: ~Q ⇒ ~P.
In words: If x is not 2, then x is not an even prime number.
This statement is true.

Exercise 10.4

1. a)

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

b)

P	Q	Q ⇒ P
T	T	T
T	F	T
F	T	F
F	F	T

c)

P	Q	~P	~Q	~P ⇒ ~Q
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

d)

P	Q	~Q	~P	~Q ⇒ ~P
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

2. The answers to a) and d) in Question 1 are the same.
3. The answers to b) and c) in Question 1 are the same.
4. a)

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∧ ~Q
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

~(P ∧ Q) and ~P ∧ ~Q do not have the same truth values, so they are not equivalent.

b)

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∨ ~Q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

~(P ∧ Q) and ~P ∨ ~Q have the same truth values, so they are equivalent.

c)

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∨ ~Q
T	T	F	F	T	F	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T

~(P ∨ Q) and ~P ∨ ~Q do not have the same truth values, so they are not equivalent.

d)

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∧ ~Q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

~(P ∨ Q) and ~P ∧ ~Q have the same truth values, so they are equivalent.

Exercise 10.5

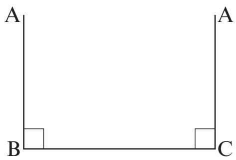
Students use an indirect proof (either method) to prove different statements. Allow them quite a bit of latitude in their proofs. The main aim is that their proofs must make sense, even if the exact methods are not followed perfectly. Suggested answers are provided on the next page.

1. Let P be the statement: 'k is the sum of two odd numbers' and Q be the statement: 'k is an even number'. We want to prove that P ⇒ Q.

Using Method 1:
Investigate the contrapositive ~Q ⇒ ~P:
~Q is the statement: 'k is an odd number', and
~P is the statement: 'k is the sum of an odd and an even number'.
If k is an odd number, it can be written as $(k - 1) + 1$, where $(k - 1)$ is an even number and 1 is an odd number.
So ~Q ⇒ ~P and therefore P ⇒ Q.

2. Let P be the statement: 'ABC is a triangle' and Q be the statement: 'ABC can have at most one right angle'. We want to prove that P ⇒ Q.

Using Method 2:
Assume that P ⇒ ~Q. So △ABC has two or more right angles, for example $\hat{B}$ and $\hat{C}$, as shown in the diagram.



AB ∥ AC (coint. ∠s are suppl.)
∴ AB and AC will never meet at a common vertex, A.
∴ ABC is not a triangle.
∴ P is true and ~P is also true, which is impossible.
∴ The assumption P ⇒ ~Q is false and hence P ⇒ Q.

3. Let P be the statement: 'ABC is an equilateral triangle' and Q be the statement: '$\hat{A} = \hat{B} = \hat{C} = 60°$'.
We want to prove that $P \Rightarrow Q$.

Using Method 1:
Investigate the contrapositive $\sim Q \Rightarrow \sim P$:
$\sim Q$ is the statement: '$\hat{A} = \hat{B} = \hat{C} \neq 60°$' and $\sim P$ is the statement: 'ABC is an equilateral triangle'.
If $\hat{A} = \hat{B} = \hat{C} \neq 60°$, then $\hat{A} + \hat{B} + \hat{C} \neq 180°$.
$\therefore$ ABC is not a triangle.
$\therefore \sim Q \Rightarrow \sim P$ and therefore $P \Rightarrow Q$.

Assess your progress

1. **a)** This is an open statement.
   b) This is a closed statement. It is false.
   c) This is a closed statement. It is true.
   d) This is a closed statement. It is true.
   e) This is a closed statement. It is false.

2. **a)** My cousin is not writing exams today.
   b) 74 is not a multiple of 4.
   c) 25 is not a perfect square.
   d) All people enjoy dancing.
   e) $0.\dot{6}$ is a rational number.

3. **a)**

P	Q	~P	~Q	~P ∨ ~Q
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

b)

P	Q	P ∨ Q	~(P ∨ Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

c)

P	Q	~Q	P ∧ Q	(P ∧ Q) ∨ ~Q
T	T	F	T	T
T	F	T	F	T
F	T	F	F	F
F	F	T	F	T

d)

P	Q	~Q	P ∨ ~Q	(P ∨ ~Q) ∧P
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	F

e)

P	Q	~P	~Q	~Q ∨ P	~P ⇒ (~Q ∨ P)
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

f)

P	Q	~Q	P ⇒ ~Q	(P ⇒ ~Q) ∨ Q
T	T	F	F	F
T	F	T	T	T
F	T	F	T	T
F	F	T	T	T

4. **a)** ~P ∨ ~Q and (P ⇒ ~Q) ∨ Q are equivalent.
b) (P ∨ ~Q) ∧ P and ~(P ∨ Q) are not equivalent.
c) (P ∧ Q) ∨ ~Q and ~P ⇒ (~Q ∨ P) are equivalent.

5. **a)** Let P be the statement 'x is a multiple of 15' and Q be the statement 'x is a multiple of 5'.
In logical notation: P ⇒ Q.
This statement is true.

b) In logical notation: Q ⇒ P.
In words: If x is a multiple of 5, then x is a multiple of 15.
This statement is false, because 10 is a multiple 5, but not of 15.

c) In logical notation: ~P ⇒ ~Q.
In words: If x is not a multiple of 15, then x is not a multiple of 5.
This statement is false, because 10 is not a multiple of 15, but it is a multiple of 5.

d) In logical notation: ~Q ⇒ ~P.
In words: If x is not a multiple of 5, then x is not a multiple of 15.

This statement is true, because any number that is not a multiple of 5 is also definitely not a multiple of 15.

6. We know that if $x^3 = 27$, then $x = 3$ and $4x = 4 \times 3 = 12$, so $P \Rightarrow Q$.
We also know if $4x = 12$ then $x = 3$ and $3^3 = 27$, so $Q \Rightarrow P$.
$P \Rightarrow Q$ and $Q \Rightarrow P$, so P and Q are equivalent statements.

7. Let P be the statement: 'k is the product of two odd numbers' and Q be the statement: 'k is an odd number'.
We want to prove that $P \Rightarrow Q$.
Using Method 1:
Investigate the contrapositive $\sim Q \Rightarrow \sim P$:
$\sim Q$ is the statement: 'k is an even number', and
$\sim P$ is the statement: 'k is not the product of two odd numbers'.
If k is an even number, it can be written as $2n$, where n is an integer. So any factor pair of k will include at least one even factor.
$\therefore \sim Q \Rightarrow \sim P$ and therefore $P \Rightarrow Q$.

TOPIC 11

Circle geometry: Chord properties

Introduction

In this topic, students will revise the properties of circles, triangles and quadrilaterals that they learnt last year.

They will focus on chords of circles, and will learn how to prove theorems and then solve problems using the facts they have proved.

Common difficulties

Students need to understand all the circle vocabulary and know the difference between a radius, a diameter and a chord.

It is also very important that they set out formal proofs of theorems properly.

Preparation

You could provide cardboard models of circles and their parts to introduce the topic.

Prepare charts showing the following, using the diagrams provided in the Student's Book:

- parts of a circle that are needed to study the theorems
- the theorem of Pythagoras
- diagrams used in the circle chord theorems.

Introduction for students

Revise the parts of a circle to make sure that all the students know the circle vocabulary, especially chords and segments.

Work through a few examples of the theorem of Pythagoras and discuss the different situations where you may need this theorem.

Remind students how to set out a formal proof with a diagram and construction lines where necessary.

Answers

Exercise 11.1

1. BC = 24 mm

2. **a)** DG = 5 cm   **b)** EF = $8\sqrt{2}$ = 11.3 cm

3. AM = 4 cm (AM = MB given)
   OM = 3 cm (Pythagoras)

4. DM = 24 cm (DM = MF)
   DO = 25 cm (Pythagoras)
   DE = 50 cm

5. MP = 50 mm (Pythagoras)
   radius = 25 mm

6. **a)** OC = $2\sqrt{3}$ (OC ∥ AD and OC = $\frac{1}{2}$ AD)
   b) $AB^2 = (4\sqrt{3})^2 + 16 = 64$
   AB = 8 cm

7. **a)** OM = 8 mm (Pythagoras)
   b) AC = 12 mm (Pythagoras)

8. **a)** $OM^2 = (ON - MN)^2$
   $= ON^2 - 2ON.MN + MN^2$ ①
   $OM^2 = OP^2 - PM^2$
   $= ON^2 - (NQ^2 - MN^2)$ ②
   $\therefore ON^2 - 2ON \,.\, MN + MN^2$
   $= ON^2 - NQ^2 + MN^2$ ① = ②
   $\therefore NQ^2 = 2ON \,.\, MN$

   b) $r = 9 = OP = ON$
   and OM = 1
   ∴ MN = 8
   $\therefore NQ^2 = 2(9)(8) = 144$
   ∴ NQ = 12 cm

Exercise 11.2

1. **a)** MQ ⊥ PR (line from circle centre bisects chord)
   and PQ = QR = 4 cm
   ∴ MQ = 3 cm (Pythagoras)

   b) radius = 17 cm (Pythagoras)

c) QR = 12 cm (Pythagoras)
$\therefore$ PR = 24 cm

2. $OM^2 = 13^2 - 5^2 = 144$ (AM = MB = 5 cm)
$\therefore$ OM = 12 cm
$\therefore$ MP = 1 cm

3. **a)** $r = x + 2$
$\therefore (x + 2)^2 = x^2 + 6^2$ (Pythagoras)
$\therefore x^2 + 4x + 4 = x^2 + 36$
$4x = 32$
$x = 8$ cm

b) $r = 10$ cm

4. ON $\perp$ CD (CN = CD)
OM $\perp$ AB (AM = MB)
$\therefore$ OC = 25 cm (Pythagoras)
$\therefore$ AM = $10\sqrt{6}$
$\therefore$ AB = $20\sqrt{6}$
AB = 49 cm

5. AC = CB = 40 cm and OC $\perp$ AB (C midpoint of chord AB)
$OA^2 = 40^2 + 40^2$
$\therefore$ OA = $40\sqrt{2}$
$\therefore$ diameter = $80\sqrt{2}$

6. **a)** AB = 24 cm (Q midpoint of chord, OQ $\perp$ AB)
b) OQ = 5 cm (Pythagoras)
$\therefore$ PQ = 8 cm (OP = OA = radii)
AP = $\sqrt{144 + 64} = 4\sqrt{13}$

7. MT = TN (given)
$\therefore$ OT $\perp$ MN (line from centre $\perp$ chord)
$\hat{N} = 45°$ (sum of $\angle$s of isosceles $\triangle$)

8. **a)** ME $\perp$ AB (given)
$\therefore$ E midpoint of AB
$\therefore$ AE = 15 cm (AE = EB)
Let DE = x
$\therefore$ CE = $5x$ (CE = 5DE)
$\therefore$ ME = $2x$ and CM = $3x$ (MC = MD radii)
$\therefore$ AM = $3x$
$\therefore AM^2 = 9x^2 = 4x^2 + 225$ (Pythagoras)

$\therefore 5x^2 = 225$
$\therefore x = \sqrt{45} = 3\sqrt{5}$
$\therefore$ DE = $3\sqrt{5}$

b) diameter of the circle = $6x = 18\sqrt{5}$

9. Let OC = x
AC = 3 = CB (line OCD $\perp$ AB)
$OB^2 = OC^2 + BC^2$
$(x + 1)^2 = x^2 + 9$
$x^2 + 2x + 1 = x^2 + 9$
$x = 4$
$\therefore$ OC = 4 cm

10. a) OP $\parallel$ MN (given)
$\therefore$ $O\hat{M}N = 90° = M\hat{O}P$ (coint. $\angle$s)
and $P\hat{N}M = 90° = O\hat{P}N$ (coint. $\angle$s)
$\therefore$ OM $\parallel$ PN (coint. $\angle$s suppl.)
$\therefore$ OPNM is a rectangle (parm with all interior $\angle$s = 90°)

b) Construct DB $\perp$ MN
$\therefore$ OD = MB and DP = BN
AM = MB (OM $\perp$ AB)
BN = NC (PN $\perp$ BC)
$\therefore$ OP = $\frac{1}{2}$ AC

Exercise 11.3

1. BC = 8 (Pythagoras)
AC = 20 (Pythagoras)
$\therefore$ AB = 20 − 8 = 12 cm

2. AE = 40 cm (OE $\perp$ AB)
CF = 30 cm (OF $\perp$ CD)
OA = OC = 50 cm (radius)
$\therefore$ OE = 30 cm (Pythagoras)
and OF = 40 cm (Pythagoras)
$\therefore$ EF = 10 cm

3. MC = 3 cm (OM $\perp$ BC)
$\therefore$ OM = 4 cm (Pythagoras)
$\therefore$ AM = 7.5 cm (Pythagoras)
$\therefore$ AD = 15 cm

4. AM = MB = 15 cm (OM ⊥ AB)
∴ OM = 20 cm (Pythagoras)
CN = ND = 20 cm (ON ⊥ CD)
∴ ON = 15 cm (Pythagoras)
∴ MN = 35 cm (MON straight line)

5. Construct D and E on ABC such that MD ⊥ AB and PE ⊥ BC.
∴ AD = DB and BE = EC
DE = MP (opp. sides in rectangle DEPM)
DE = DB + BE
$= \frac{1}{2}(AB + BC)$
$= \frac{1}{2}AC$
∴ AC = 2MP

6. MR = RN = 40 cm (OR ⊥ MN)
PS = 26 = SQ (OS ⊥ PQ)
ON = 60 = OP (radius)
∴ OR = $\sqrt{2\,000}$ (Pythagoras)
∴ OS = $\sqrt{2\,924}$ (Pythagoras)
∴ RS = $\sqrt{2\,924} - \sqrt{2\,000}$
= 9.35 cm

Assess your progress

1. **a)** PM = MQ = 40 (M midpoint of chord PQ)
and PM ⊥ MO
∴ OP = 50 cm (Pythagoras)

b) OR = 50 (radius)
RT = $\sqrt{2\,304}$ (Pythagoras)
RT = 48 cm
∴ RS = 96 cm

2. GJ = 18 = JH (J midpoint of chord GH)
let OJ = x
OK = OG = $x + 6$
$(x + 6)^2 = x^2 + 18^2$ (Pythagoras)
∴ $x^2 + 12x + 36 = x^2 + 18^2$
∴ $12x = 288$
∴ $x = 24$ cm
∴ OJ = 24 cm

3. **a)** AM = 40 cm (Pythagoras)
b) AM = MB ($OM \perp AB$)
$\therefore$ AB = 80 cm

4. $OM \perp CD$ (M midpoint)
CM = 20 = MD (M midpoint)
$OD = \sqrt{800} = 20\sqrt{2}$
$DE = 40\sqrt{2} = 56.57$ cm

5. OG = 50 cm (radius = $\frac{1}{2}$diameter)
$\therefore$ MG = 30 cm (Pythagoras)
and MG = MF ($OM \perp FG$)
$\therefore$ FG = 60 cm

6. **a)** AE = 12 = AD ($OA \perp DE$)
$\therefore$ OA = 5 cm (Pythagoras)
and OB = 12 cm (Pythagoras)
$\therefore$ AB = 7 cm
b) BC = 1 cm (OC radius)

7. **a)** $OB^2 = (x + 2)^2 + (3x + 3)^2$
and $OA^2 = (x + 2)^2 + (5x - 3)^2$
$\therefore (3x + 3)^2 = (5x - 3)^2$
$9x^2 + 18x + 9 = 25x^2 - 30x + 9$
$x = 3$
$AB = 8x = 24$ units
b) radius $OA = \sqrt{169} = 13$ (Pythagoras)
$\therefore$ CD = 13 − 5 = 8 units

8. **a)** PS = SQ ($OS \perp PQ$)
$\therefore$ PQ = 65 cm
b) PT = 60 (Pythagoras in $\triangle PQT$)
$\therefore$ PR = 65 cm (Pythagoras in $\triangle PRT$)

9. MC = DB = 13 cm (diagonals of a rectangle)
$\therefore$ DM = 5 cm
$\therefore$ BM = 12 cm (Pythagoras)
$\therefore$ Area = $5 \times 12 = 60$ cm^2

10. **a)** T **b)** T **c)** F **d)** T

TOPIC 12 Circle geometry: Angle properties

Introduction

In this topic students will learn more about the properties of angles that lie inside a circle, and will prove theorems involving angle properties.

They will then move on to learning about the properties of cyclic quadrilaterals, and will also prove theorems involving these properties.

Students will solve problems using the facts they have learnt about angles within circles and cyclic quadrilaterals.

Common difficulties

Students need to understand all the circle vocabulary and know the vocabulary for different sized angles. They need to know how to set out formal proofs of theorems properly.

Some students confuse the meanings of complementary angles which add up to 90° and supplementary angles which add up to 180°.

Students are sometimes confused between a corollary and a converse. Remind them that a corollary is a fact, without a proof, that follows from a theorem.

However, a converse is the opposite of what has been proved in a theorem. For example we can say 'If two triangles are congruent then they are similar', and the converse then states 'If two triangles are similar, then they are congruent'. This converse statement is not true in this case. Converses to theorems are not necessarily always true so we need to have a formal proof to show when a converse to a statement is true.

Preparation

Have diagrams of the circle angle theorems and cyclic quad theorems on the walls of the classroom so that students become familiar with these.

Make a chart of the three ways to prove a quadrilateral is cyclic as these are used many times when solving problems.

Introduction for students

Revise the aspects of circle geometry discussed under 'Common difficulties' above.

Remind students that in formal proofs, in riders and in calculations in geometry, they need to provide reasons for facts used. They can use brackets when they are writing in the reasons, although in the Student's Book brackets are not used. It helps with reading if the reasons are indented away from the written proofs and calculations, but this is not obligatory. Go through the conventions for using abbreviations and symbols within reasons, as shown in the Student's Book and this Teacher's Guide.

Answers

Exercise 12.1

1. $a = 90°$ (∠ subtended by diameter)
$b = 80°$ (∠ at centre = twice ∠ on circumf.)
$c = 40°$ (∠ on straight line CBD)
$d = 260°$ (∠ at centre = twice ∠ on circumf.)
$e = 25°$ (alt. ∠s, ∥ lines)
$f = 94°$ (∠ at centre = twice ∠ on circumf.)
$g = 130°$ (∠ at centre = twice ∠ on circumf.)
$h = 43°$ (sum of ∠s of a △)
$i = 55°$ (∠ at centre = twice ∠ on circumf.)
$j = 30°$ (∠ at centre = twice ∠ on circumf.)
$k = 40°$ (exterior ∠ of △ = sum int. opp. ∠s)
$l = 10°$ (∠s in the same segment)

2. $p = q = 55°$ (equal ∠s in isosceles △)
$r = s = 55°$ (equal ∠s in isosceles △)

3. $\hat{Q} = 35°$ (PQ ∥ SR)
$y = 35°$ (∠s in same segment)
$x = 68°$ (sum of ∠s of △PSR)

4. $a = 30°$ (base ∠ in isosceles △)
$b = 60°$ (equilateral △OQR)
$c = 120°$ (∠ on straight line)
$d = 36°$ (∠s in same segment)

$e = 66°$ (sum of ∠s of △ABE)
$f = 78°$ (vert. opp. ∠s)
$g = 66°$ (∠s in same segment)
$h = 24°$ (∠s on equal chords)
$i = 36°$ (∠s on equal chords)
$j = 50°$ (∠s in same segment)
$k = 28°$ (∠s in same segment)
$l = 50°$ (exterior ∠ of △ = sum int. opp. ∠s)
$m = 40°$ (exterior ∠ of △ = sum int. opp. ∠s)
$n = 40°$ (sum of ∠s of △)
$o = 48°$ (∠s in isosceles △)
$p = 48°$ (∠s in same segment)
$q = 32°$ (∠s on equal chords)
$r = 84°$ (∠ at centre = twice ∠ on circumf.)
$s = 48°$ (∠ in isosceles △)
$t = 30°$ (vert. opp. ∠s)
$u = 30°$ (∠s in same segment)
$v = 70°$ (sum of ∠s of △BCD)
$w = 48°$ (∠ at centre = twice ∠ on circumf.)

Exercise 12.2

1. $4a + a = 180°$ (opp. ∠s in cyclic quad)
$\therefore a = 36°$
$\therefore b = 72°$(∠ at centre = twice ∠ on circumf.)
$c = 30°$(∠s on a straight line)
$d = 75°$(base ∠ − isosceles △)
$e = 37.5°$(base ∠ − isosceles △)
$4f = 180°$, $\therefore f = 45°$
$3g = 180°$, $\therefore g = 60°$
$h = 70°$ (∠ at centre = twice ∠ on circumf.)
$i = 110°$ (opp. ∠s in cyclic quad)
$j = 105°$ (ext. ∠ = sum int. opp. ∠s)
$k = 86°$ (opp. ∠s in cyclic quad)
$l = 42°$ (base ∠ − isosceles △)
$m = 92°$ (opp. ∠s in cyclic quad)
$n = 60°$ (opp. ∠s in cyclic quad)
$p = 155°$ (∠ at centre = twice ∠ on circumf.)
$q = 94°$ (corresp. ∠s, ∥ lines)
$r = 90°$ (∠ in semi-circle)
$s = 142°$ (opp. ∠s in cyclic quad)
$t = 52°$ (sum of ∠s of a △)

2. $a = 50°$ (∠ at centre = twice ∠ on circumf.)
$b = 65°$ (sum of ∠s of a △)
$c = 40°$ (sum of ∠s of a △)

3. $x = 168°$ (∠ at centre = twice ∠ on circumf.)
$y = 12°$ (coint. ∠s, ∥ lines)
$z = 96°$ (ext. ∠ cyclic quad)

4. **a)** In △s VST and VSQ

TV = VQ	(OV ⊥ TQ, given)
VS is common	
$T\hat{V}S = 90° = S\hat{V}Q$	
∴ △VST ≡ △VSQ	(SAS)
$\therefore \hat{S}_1 = \hat{S}_2$	

b)

$Q\hat{R}T = Q\hat{S}T$	(equal ∠s in same segment)
$\hat{R} = \hat{S}_{1+2} = 2\hat{S}_2$	

c)

$Q\hat{R}T = Q\hat{S}T$	(proved above)
∴ QRST is a cyclic quad	(equal ∠s in same segment)

5. **a)**

$D\hat{E}C = x = \hat{A}$	(corresp. ∠s, CE ∥ AB)
$D\hat{C}E = x$	(ext. ∠ cyclic quad)
∴ △CDE is isosceles	(base ∠s equal)

b)

$E\hat{O}B = 2x$	(∠ at centre = twice ∠ on circumf.)
$\hat{D} = 180° - 2x$	(sum of ∠s of △)
∴ DEOB is a cyclic quad	(opp. ∠s suppl.)

Assess your progress

1.

OA = 5	(radius)
∴ AB = 3	(Pythagoras)
∴ AC = 6 units	(AB = BC)

2.

OE = 5	(Pythagoras)
$x = 2$ units	(radius)

$y = \sqrt{8^2 + 4^2} = 4\sqrt{5} = 8.94$ units

3.

OQ = 8 units	(radius)
∴ PQ = 6	(Pythagoras)
∴ PR = 12	(OQ ⊥ PR)

4. $a = 62°$ (vert. opp. ∠s)
$b = 76°$ (∠s in same segment)

$c = 42°$ (∠s in same segment)
$d = 105°$ (∠s on straight line)
$e = 94°$ (opp. ∠s cyclic quad)
$f = 75°$ (ext. ∠ cyclic quad)
$g = 75°$ (vert. opp. ∠s)

5. $a = 43°$ (∠s in same segment)
$b = 86°$ (∠ in centre = twice ∠ on circumf.)
$c = 117°$ (opp. ∠s cyclic quad)
$d = 90°$ (ABCD kite)
$e = 90°$ (ABCD kite)
$f = 20°$ (∠s on equal chords)

6. $a = 125°$ (ext. ∠ cyclic quad)
$b = 70°$ (opp. ∠s cyclic quad ABCD)
$c = 95°$ (opp. ∠s cyclic quad ACDE)
$d = 60°$ (opp. ∠s cyclic quad BDEF)
$e = 60°$ (alt. ∠s AE ∥ BD)
$f = 95°$ (opp. ∠s cyclic quad BFED)
$g = 85°$ (coint. ∠s AE ∥ BD)
$h = 84°$ (ext. ∠ isosceles △)
$i = 96°$ (opp. ∠s cyclic quad ABCD)
$j = 48°$ (∠ in centre = twice ∠ on circumf.)
$k = 42°$ (base ∠s isosceles △)
$l = 106°$ (opp. ∠s cyclic quad ABCD)

7. **a)** $A\hat{O}C = 96°$ (∠ in centre = twice ∠ on circumf.)
b) $\hat{D}_1 = 48°$ (ext. ∠ cyclic quad)
c) $\hat{F} = 48°$ (∠s in same segment)
d) $\hat{A}_4 = 48°$ (sum of ∠s of △ADE)

8. **a)** $\hat{A}_{1+2} = 60°$ (∠ at centre = twice ∠ at circumf.)
$\hat{A}_1 = 30°$ ($\hat{A}_1 = \hat{A}_2$, given)
b) $A\hat{C}B = 30°$ (alt. ∠s, AF ∥ BC)
c) $A\hat{C}B = 30° = A\hat{D}B$ (∠s in same segment)
EA = ED (base ∠s in △AED are equal)

9. $A\hat{B}D = 50°$ (ext. ∠ = sum int. opp. ∠s)
∴ $A\hat{O}D = 100°$ (∠ at centre = twice ∠ at circumf.)

10. $A\hat{D}B = x$ (∠s in same segment)
$D\hat{A}B = x$ (ext. ∠ cyclic quad)
∴ BD = AB (base ∠s in △ABD are equal)

TOPIC 13

Circle geometry: Tangents

Introduction

The following work is covered in this topic:
- properties of tangents to a circle
- proofs of tangent theorems
- application of properties of tangents to circles in solving problems.

Common difficulties

Students are sometimes confused between a secant, a chord and a tangent. Use the chart given below to remind them of the meanings of the terms.

In this section, students will need to know the theorem of Pythagoras to calculate lengths.

Preparation

Make a chart showing the different parts of a circle that are needed to study the circle geometry theorems.

Have diagrams of the circle tan–chord theorems on the walls of the classroom so that students become familiar with these.

Introduction for students

Revise the parts of a circle learnt so far to make sure that all the students know the circle vocabulary. Explain the difference between a chord and a secant and a tangent. Check whether they remember that a diameter or radius is perpendicular to a tangent line at the point of contact. This fact results from the line from the centre perpendicular to a chord theorem.

Remind students how to set out a formal proof, using a diagram and constructions where necessary.

Remind students also to provide reasons in their proofs, riders and calculations.

Answers

Exercise 13.1

1. OB ⊥ DC (radius ⊥ tangent)
∴ OB = 8 cm (Pythagoras)
∴ diameter AB = 16 cm

2. OQ ⊥ PQ (radius ⊥ tangent)
∴ OP = 25 cm (Pythagoras)

3. $a = 40°$ $d = 118°$ $f = 36°$
$b = 10°$ $5e = 90°$
$c = 62°$ $\therefore e = 18°$

4. **a)** $\hat{O}_1 = 64°$ **b)** $\hat{D} = 32°$

5. $O\hat{B}E = 42°$ (compl. ∠, radius ⊥ tangent)
$\hat{E} = 90°$ (∠ in a semi-circle)
$\hat{D} = 48°$ (sum of ∠s of a △)

Exercise 13.2

1. $a = 52°$ $5d = 180°$ $g = 62°$
$b = 72°$ $d = 36°$ $h = 59°$
$4c = 90°$ $e = 82°$
$\therefore c = 22.5°$ $f = 98°$

2. **a)** $x = 137°$ **b)** $x = 38°$
c) $x = 40°$ and $y = 70°$ **d)** $x = 27.5°$ and $y = 35°$
e) $x = 52°$ and $y = 26°$ **f)** $x = 106°$ and $y = 37°$
g) $x = 14°$ and $y = 40°$ **h)** $x = 28°$ and $y = 56°$

3. **a)** $C\hat{D}E = 30°$ **b)** $A\hat{B}C = 90°$ **c)** $B\hat{A}D = 60°$

4. **a)** $B\hat{E}D = x$ **b)** $A\hat{B}E = 2x$ **c)** $E\hat{B}O = 90° - 2x$

Exercise 13.3

1. $O\hat{B}A = 90°$ (radius ⊥ tangent)
$O\hat{C}A = 90°$ (radius ⊥ tangent)
∴ ABOC is a cyclic quad (opp. ∠s suppl.)

2. $\hat{B} = 90°$ (∠ in a semicircle)
$\therefore C\hat{A}B = 45°$ (base ∠ isosceles △)

$\therefore C\hat{A}P = 90°$ (given $B\hat{A}P = 45°$)
$\therefore$ AP is a tangent to the circle (radius $\perp$ tangent)

3. **a)** $A\hat{C}E = x$ (tan FE chord AE)
$\therefore E\hat{A}C = 180° - 2x$ (sum of $\angle$s of isosceles $\triangle$)
$\therefore B\hat{C}A = 180° - 2x$ (alt. $\angle$s, $\parallel$ lines)
$\therefore E\hat{C}D = x$ ($\angle$s on straight line BCD)

b) $C\hat{E}A = x$ (alt. $\angle$s, $\parallel$ lines)
$\therefore A\hat{B}C = 180° - x$ (opp. $\angle$s in cyclic quad)

c) $E\hat{A}C = y$ (tan DE chord EC)
$B\hat{C}A = y$ (alt. $\angle$s, $\parallel$ lines)
$B\hat{A}C = y$ (base $\angle$s isosceles $\triangle$)

d) $2y + (180° - x) = 180°$
$\therefore 2y = x$
also: $y = 180° - 2x$
$y = 180° - 2(2y)$
$5y = 180°$
$\therefore y = 36°$
$\therefore x = 72°$

4. **a)** $D\hat{A}C = 30°$ (tangent AD chord AC)
$\therefore O\hat{A}C = 60°$ (radius OA $\perp$ tangent AD)
$\therefore O\hat{C}A = 60°$ (base $\angle$s isosceles $\triangle OAC$)
$\therefore \triangle AOC$ is an equilateral $\triangle$

b) $\hat{D} = 30°$ (ext. $\angle$ = sum int. opp. $\angle$s)
$\therefore AC = CD$ (isosceles $\triangle$)
$\therefore OC = CD$ ($OA = AC$)

5. CDE is a tangent to the circle. AE $\parallel$ BD
a) $\hat{A} = \hat{B}_1$ (corresp. $\angle$s AE $\parallel$ BD)
and $\hat{A} = \hat{E}_1$ (tan ED chord BE)
$\therefore \hat{E}_1 = \hat{B}_1$
b) $\hat{E}_2 = \hat{B}_2$ (alt. $\angle$s AE $\parallel$ BD)
$\therefore \triangle AEB \parallel\!| \triangle EBD$ (AAA)
$D_2 = B_3$

Assess your progress

1.

$a = 65°$	$d = 130°$	$g = 46°$	$j = 28°$
$b = 55°$	$e = 65°$	$h = 46°$	$k = 28°$
$c = 65°$	$f = 50°$	$i = 44°$	$l = 28°$

2.	$\hat{B} = 7m$	(alt. ∠s ∥ lines)
	and $\hat{A} = 7m$	(tangent CE chord CB)
	$\therefore 20m = 180°$	(sum ∠s in △)
	$m = 9°$	
3.	$F\hat{E}B = 40°$	(tangent AB chord FB)
	$\therefore x = 84°$	(sum ∠s in △)
	$\therefore y = 96°$	(opp. ∠s in a cyclic quad)
4.	**a)** $F\hat{D}B = 40°$	(tangent AB chord FB)
	and $F\hat{B}D = 90°$	(∠ in semicircle)
	$\therefore \hat{F} = 50°$	(sum ∠s in △)
	$\therefore$ and $C\hat{B}D = 50°$	(∠s on a straight line)
	$\therefore FD \parallel BC$	(alt. ∠s are equal)
	b) $F\hat{D}B = 40° = \hat{E}$	(base ∠s isosceles △)
5.	**a)** $\hat{T}_2 = 90°$	(∠ in semicircle)
	$\therefore \hat{T}_{3+4} = 90°$	(∠s on straight line)
	$\hat{R} = 90°$	(given)
	$\therefore$ RSTQ is a cyclic quad	(opp. ∠s supplementary)
	b) $\hat{S}_1 = \hat{T}_3$	(∠s in same segment)
	and $\hat{T}_3 = \hat{P}$	(tangent VTR chord TQ)
	$\therefore \hat{S}_1 = \hat{P}$	
	c) $\hat{Q}_{2+3} = 90° + \hat{P}$	(ext. ∠ of △)
	$\therefore \hat{S}_{1+2} = 90° - \hat{P}$	(opp. ∠s cyclic quad)
	and $\hat{T}_4 = 90° - \hat{P}$	
	RT = RS	(base ∠s are equal)
	d) $\hat{S}_1 = \hat{P}$	(proved above)
	$\therefore \hat{Q}_3 = 90° - \hat{P}$	(sum of ∠s of △RQS)
	and $\hat{Q}_1 = 90° - \hat{P}$	(sum of ∠s of △PQT)
	$\therefore \hat{Q}_1 = \hat{Q}_3$	

TOPIC 14 Trigonometry

Introduction

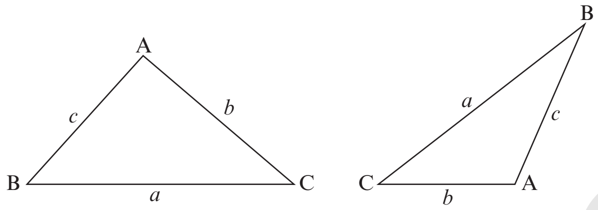
In JSS3 and SS1 your students worked with trigonometry in right-angled triangles. In SS2 they will extend their knowledge to include the trigonometry of triangles that are not necessarily right-angled. In particular, they will learn how to use the sine and cosine rules to solve a variety of trigonometric problems.

Common difficulties in this topic

- When solving triangles, units of length need only to be written in the final answer. If no units of length are given, the students should write a number only, or a number followed by 'units'.
- When students sketch their own diagrams as an aid to solving a problem, the diagrams do not need to be exactly according to scale, but it will be more helpful if they are at least more or less according to scale. For example, if an angle is given as 120°, then that angle should be drawn as an obtuse angle. If one side is longer than another, this should ideally be shown in the diagram as well.
- If the students get an error on their calculators when using $\sin^{-1}$ or $\cos^{-1}$ to calculate an angle, it is likely that the value that they are entering is greater than 1 or smaller than −1. This is of course undefined, so the students need to check their work carefully to find the cause of the problem.
- Remind your students to make sure **always** that their calculators are in Degree mode when they do trigonometric calculations.
- Rounding errors are common in trigonometric calculations. Check to see why these are happening. Do your students know how to round off correctly to a given number of decimal places? In a multi-step calculation, are they rounding intermediate answers off? Advise them to round off their final answers only.

Preparation

Prepare a chart showing the sine rule and the cosine rule for a scalene (all sides different lengths) acute-angled triangle as well as for a scalene obtuse-angled triangle. Display this chart in your class.



Introduction for students

Briefly revise the trigonometry that your students learnt in SS1. Make sure that they remember and understand the definitions of the three basic trigonometric ratios, and that they understand the CAST diagram, in particular the 2nd quadrant.

In SS2 they will be working with obtuse angles as well as acute angles, and they need to be aware that:

- $\sin (180 - \theta) = \sin \theta$
- $\cos (180 - \theta) = -\cos \theta$
- $\tan (180 - \theta) = -\tan \theta$

Finally, refer your students to the trigonometric tables on pages 312 to 323 of the Student's Book. They have already worked with these tables in SS1. Check to make sure that they remember how to use each table.

Answers

Exercise 14.1

1. $\hat{A} = 180° - 50° - 47°$ (sum of $\angle$s of a $\triangle$)

$= 83°$

$\frac{AB}{\sin C} = \frac{AC}{\sin B}$ (sine rule)

$\therefore \frac{AB}{\sin 47°} = \frac{25}{\sin 50°}$

$\therefore AB = \frac{25 \sin 47°}{\sin 50°}$

$= 23.87$ units

$\frac{BC}{\sin A} = \frac{AC}{\sin B}$ (sine rule)

$\therefore \frac{BC}{\sin 83°} = \frac{25}{\sin 50°}$

$\therefore BC = \frac{25 \sin 83°}{\sin 50°}$

$= 32.88$ units

2. $\frac{\sin F}{DE} = \frac{\sin D}{EF}$ (sine rule)

$\therefore \frac{\sin F}{3} = \frac{\sin 39°}{6}$

$\therefore \sin F = \frac{3 \sin 39°}{6}$

$\therefore \hat{F} = 18.4°$

$\hat{E} = 180° - 39° - 18.4°$ (sum of $\angle$s of a $\triangle$)

$= 122.6°$

$\frac{DF}{\sin E} = \frac{EF}{\sin D}$ (sine rule)

$\therefore \frac{DF}{\sin 122.6°} = \frac{6}{\sin 39°}$

$\therefore DF = \frac{6 \sin 122.6°}{\sin 39°}$

$= 8.0$ cm

3. **a)** $\hat{L} = 180° - 50° - 55°$ (sum of $\angle$s of a $\triangle$)

$= 75°$

$\frac{LN}{\sin M} = \frac{LM}{\sin N}$ (sine rule)

$\therefore \frac{LN}{\sin 50°} = \frac{13}{\sin 55°}$

$\therefore LN = \frac{13 \sin 50°}{\sin 55°}$

$= 12.16$ units

$\frac{MN}{\sin L} = \frac{LM}{\sin N}$ (sine rule)

$\therefore \frac{MN}{\sin 75°} = \frac{13}{\sin 55°}$

$\therefore MN = \frac{13 \sin 75°}{\sin 55°}$

$= 15.33$ mm

b) $\frac{\sin U}{ST} = \frac{\sin S}{TU}$ (sine rule)

$\therefore \frac{\sin U}{148} = \frac{\sin 88°}{326}$

$\therefore \sin U = \frac{148 \sin 88°}{326}$

$\therefore \hat{U} = 27.0°$

$\hat{T} = 180° - 88° - 27°$ (sum of $\angle$s of a $\triangle$)

$= 65°$

$\frac{SU}{\sin T} = \frac{TU}{\sin S}$ (sine rule)

$\therefore \frac{SU}{\sin 65°} = \frac{326}{\sin 88°}$

$\therefore SU = \frac{326 \sin 65°}{\sin 88°}$

$= 295.64$ cm

c) $\hat{R} = \hat{Q}$ (isosceles △)

$= 30°$

$\hat{P} = 180° - 30° - 30°$ (sum of ∠s of a △)

$= 120°$

$\frac{PR}{\sin Q} = \frac{QR}{\sin P}$ (sine rule)

$\therefore \frac{PR}{\sin 30°} = \frac{75}{\sin 120°}$

$\therefore PR = \frac{75 \sin 30°}{\sin 120°}$

$= 43.30$ m

$\therefore PQ = 43.30$ m (isosceles △)

d) $\frac{\sin B}{AC} = \frac{\sin C}{AB}$ (sine rule)

$\therefore \frac{\sin B}{4} = \frac{\sin 93°}{5.1}$

$\therefore \sin B = \frac{4 \sin 93°}{5.1}$

$\therefore \hat{B} = 51.6°$

$\hat{A} = 180° - 93° - 51.6°$ (sum of ∠s of a △)

$= 35.4°$

$\frac{BC}{\sin A} = \frac{AB}{\sin C}$ (sine rule)

$\therefore \frac{BC}{\sin 35.4°} = \frac{5.1}{\sin 93°}$

$\therefore BC = \frac{5.1 \sin 35.4°}{\sin 93°}$

$= 2.96$ km

4. a) $\hat{C} = 180° - 105° - 22°$ (sum of ∠s of a △)

$= 53°$

$\frac{AC}{\sin B} = \frac{AB}{\sin C}$ (sine rule)

$\therefore \frac{AC}{\sin 22°} = \frac{19}{\sin 53°}$

$\therefore AC = \frac{19 \sin 22°}{\sin 53°}$

$= 8.91$cm

$\frac{BC}{\sin A} = \frac{AB}{\sin C}$ (sine rule)

$\therefore \frac{BC}{\sin 105°} = \frac{19}{\sin 53°}$

$\therefore BC = \frac{19 \sin 105°}{\sin 53°}$

$= 22.98$ cm

b) $\frac{\sin D}{EF} = \frac{\sin F}{DE}$ (sine rule)

$\therefore \frac{\sin D}{65} = \frac{\sin 58°}{72}$

$\therefore \sin D = \frac{65 \sin 58°}{72}$

$\therefore \hat{D} = 50.0°$

$\hat{E} = 180° - 58° - 50°$ (sum of ∠s of a △)

$= 72°$

$\frac{DF}{\sin E} = \frac{DE}{\sin F}$ (sine rule)

$\therefore \frac{DF}{\sin 72°} = \frac{72}{\sin 58°}$

$\therefore DF = \frac{72 \sin 72°}{\sin 58°}$

$= 80.75$ m

c) $\hat{L} = 180° - 79° - 57°$ (sum of ∠s of a △)

$= 44°$

$\frac{JL}{\sin K} = \frac{KL}{\sin J}$ (sine rule)

$\therefore \frac{JL}{\sin 57°} = \frac{5.5}{\sin 79°}$

$\therefore JL = \frac{5.5 \sin 57°}{\sin 79°}$

$= 4.70$ units

$\frac{JK}{\sin L} = \frac{KL}{\sin J}$ (sine rule)

$\therefore \frac{JK}{\sin 44°} = \frac{5.5}{\sin 79°}$

$\therefore JK = \frac{5.5 \sin 44°}{\sin 79°}$

$= 3.89$ units

d) $\frac{\sin O}{MN} = \frac{\sin N}{MO}$ (sine rule)

$\therefore \frac{\sin O}{3} = \frac{\sin 124°}{8}$

$\therefore \sin O = \frac{3 \sin 124°}{8}$

$\therefore \hat{O} = 18.1°$

$\hat{M} = 180° - 124° - 18.1°$ (sum of ∠s of a △)

$= 37.9°$

$\frac{NO}{\sin M} = \frac{MO}{\sin N}$ (sine rule)

$\therefore \frac{NO}{\sin 37.9°} = \frac{8}{\sin 124°}$

$\therefore NO = \frac{8 \sin 37.9°}{\sin 124°}$

$= 5.935$ mm

e) $\hat{P} = 180° - 37.2° - 34.5°$ (sum of ∠s of a △)

$= 108.3°$

$$\frac{PQ}{\sin R} = \frac{PR}{\sin Q} \qquad \text{(sine rule)}$$

$$\therefore \frac{PQ}{\sin 37.2°} = \frac{73}{\sin 34.5°}$$

$$\therefore PQ = \frac{73 \sin 37.2°}{\sin 34.5°}$$

$$= 77.92 \text{ m}$$

$$\frac{QR}{\sin P} = \frac{PR}{\sin Q} \qquad \text{(sine rule)}$$

$$\therefore \frac{QR}{\sin 108.3°} = \frac{73}{\sin 34.5°}$$

$$\therefore QR = \frac{73 \sin 108.3°}{\sin 34.5°}$$

$$= 122.36 \text{ m}$$

f)

$$\frac{\sin X}{YZ} = \frac{\sin Z}{XY} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin X}{5} = \frac{\sin 60.4°}{7}$$

$$\therefore \sin X = \frac{5 \sin 60.4°}{7}$$

$$\therefore \hat{X} = 38.4°$$

$$\hat{Y} = 180° - 60.4° - 38.4° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 81.2°$$

$$\frac{XZ}{\sin Y} = \frac{XY}{\sin Z} \qquad \text{(sine rule)}$$

$$\therefore \frac{XZ}{\sin 81.2°} = \frac{7}{\sin 60.4°}$$

$$\therefore XZ = \frac{7 \sin 81.2°}{\sin 60.4°}$$

$$= 7.96 \text{ units}$$

Exercise 14.2

1. a)

$$\frac{\sin C}{AB} = \frac{\sin B}{AC} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin C}{4} = \frac{\sin 58°}{3.5}$$

$$\therefore \sin C = \frac{4 \sin 58°}{3.5}$$

$$\therefore \hat{C} = 75.7° \text{ OR } \hat{C} = 180° - 75.7°$$

$$= 104.3°$$

First solution, using $\hat{C} = 75.7°$:

$$\hat{A} = 180° - 58° - 75.7° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 46.3°$$

$$\frac{BC}{\sin A} = \frac{AC}{\sin B} \qquad \text{(sine rule)}$$

$$\therefore \frac{BC}{\sin 46.3°} = \frac{3.5}{\sin 58°}$$

$$\therefore BC = \frac{3.5 \sin 46.3°}{\sin 58°}$$

$$= 2.98 \text{ cm}$$

Second solution, using $\hat{C} = 104.3°$:

$\hat{A} = 180° - 58° - 104.3°$ (sum of ∠s of a △)

$= 17.7°$

$\frac{BC}{\sin A} = \frac{AC}{\sin B}$ (sine rule)

$\therefore \frac{BC}{\sin 17.7°} = \frac{3.5}{\sin 58°}$

$\therefore BC = \frac{3.5 \sin 17.7°}{\sin 58°}$

$= 1.25$ cm

b) $\frac{\sin E}{DF} = \frac{\sin D}{EF}$ (sine rule)

$\therefore \frac{\sin E}{43} = \frac{\sin 76.1}{42}$

$\therefore \sin E = \frac{43 \sin 76.1}{42}$

$\therefore \hat{E} = 83.6°$ OR $\hat{E} = 180° - 83.6°$

$= 96.4°$

First solution, using $\hat{E} = 83.6°$:

$\hat{F} = 180° - 76.1° - 83.6°$ (sum of ∠s of a △)

$= 20.3°$

$\frac{DE}{\sin F} = \frac{EF}{\sin D}$ (sine rule)

$\therefore \frac{DE}{\sin 20.3°} = \frac{36}{\sin 76.1°}$

$\therefore DE = \frac{36 \sin 20.3°}{\sin 76.1°}$

$= 12.87$ mm

Second solution, using $\hat{E} = 96.4°$:

$\hat{F} = 180° - 76.1° - 96.4°$ (sum of ∠s of a △)

$= 7.5°$

$\frac{DE}{\sin F} = \frac{EF}{\sin D}$ (sine rule)

$\therefore \frac{DE}{\sin 7.5°} = \frac{36}{\sin 76.1°}$

$\therefore DE = \frac{36 \sin 7.5°}{\sin 76.1°}$

$= 4.84$ mm

c) $\frac{\sin J}{KL} = \frac{\sin K}{JL}$ (sine rule)

$\therefore \frac{\sin J}{65} = \frac{\sin 44°}{50}$

$\therefore \sin J = \frac{65 \sin 44°}{50}$

$\therefore \hat{J} = 64.6°$ OR $\hat{J} = 180° - 64.6°$

$= 115.4°$

First solution, using $\hat{J} = 64.6°$:

$\hat{L} = 180° - 44° - 64.6°$ (sum of ∠s of a △)

$= 71.4°$

$$\frac{JK}{\sin L} = \frac{JL}{\sin K} \qquad \text{(sine rule)}$$

$$\therefore \frac{JK}{\sin 71.4°} = \frac{50}{\sin 44°}$$

$$\therefore JK = \frac{50 \sin 71.4°}{\sin 44°}$$

$$= 68.22 \text{ units}$$

Second solution, using $\hat{J} = 115.4°$:

$$\hat{L} = 180° - 44° - 115.4° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 20.6°$$

$$\frac{JK}{\sin L} = \frac{JL}{\sin K} \qquad \text{(sine rule)}$$

$$\therefore \frac{JK}{\sin 20.6°} = \frac{50}{\sin 44°}$$

$$\therefore JK = \frac{50 \sin 20.6°}{\sin 44°}$$

$$= 25.32 \text{ units}$$

d) $$\frac{\sin Q}{PR} = \frac{\sin R}{PQ} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin Q}{19} = \frac{\sin 41°}{17}$$

$$\therefore \sin Q = \frac{19 \sin 41°}{17}$$

$$\therefore \hat{Q} = 47.2° \text{ OR } \hat{Q} = 180° - 47.2°$$

$$= 132.8°$$

First solution, using $\hat{Q} = 47.2°$:

$$\hat{P} = 180° - 41° - 47.2° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 91.8°$$

$$\frac{QR}{\sin P} = \frac{PQ}{\sin R} \qquad \text{(sine rule)}$$

$$\therefore \frac{QR}{\sin 91.8°} = \frac{17}{\sin 41°}$$

$$\therefore QR = \frac{17 \sin 91.8°}{\sin 41°}$$

$$= 25.90 \text{ m}$$

Second solution, using $\hat{Q} = 132.8°$:

$$\hat{P} = 180° - 41° - 132.8° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 6.2°$$

$$\frac{QR}{\sin P} = \frac{PQ}{\sin R} \qquad \text{(sine rule)}$$

$$\therefore \frac{QR}{\sin 6.2°} = \frac{17}{\sin 41°}$$

$$\therefore QR = \frac{17 \sin 6.2°}{\sin 41°}$$

$$= 2.80 \text{ m}$$

2. a) $\frac{\sin Y}{XZ} = \frac{\sin X}{YZ}$ (sine rule)

$\therefore \frac{\sin Y}{6.9} = \frac{\sin 25°}{4.8}$

$\therefore \sin Y = \frac{6.9 \sin 25°}{4.8}$

$\therefore \hat{Y} = 37.4°$ OR $\hat{Y} = 180° - 37.4°$

$= 142.6°$

b) $\frac{\sin Y}{XZ} = \frac{\sin Z}{XY}$ (sine rule)

$\therefore \frac{\sin Y}{\sqrt{50}} = \frac{\sin 45°}{5}$

$\therefore \sin Y = \frac{\sqrt{50} \sin 45°}{5}$

$\therefore \hat{Y} = 90°$

c) $\frac{\sin Y}{XZ} = \frac{\sin X}{YZ}$ (sine rule)

$\therefore \frac{\sin Y}{17.94} = \frac{\sin 30°}{8.97}$

$\therefore \sin Y = \frac{17.94 \sin 30°}{8.97}$

$\therefore \hat{Y} = 90°$

d) $\frac{\sin X}{YZ} = \frac{\sin Z}{XY}$ (sine rule)

$\therefore \frac{\sin X}{80} = \frac{\sin 50°}{64}$

$\therefore \sin X = \frac{80 \sin 50°}{64}$

$\therefore \hat{X} = 73.2°$ OR $\hat{X} = 180° - 73.2°$

$= 106.8°$

$\therefore \hat{Y} = 180° - 50° - 73.2°$ (sum of $\angle$s of a $\triangle$)

$= 56.8°$

OR $\hat{Y} = 180° - 50° - 106.8°$ (sum of $\angle$s of a $\triangle$)

$= 23.2°$

Exercise 14.3

1. a) $b^2 = a^2 + c^2 - 2ac \cos B$ (cosine rule)

$\therefore AC^2 = 38^2 + 46^2 - 2(38)(46) \cos 57°$

$\therefore AC = \sqrt{38^2 + 46^2 - 2(38)(46) \cos 57°}$

$= 40.69$ units

$\frac{\sin C}{c} = \frac{\sin B}{b}$ (sine rule)

$\therefore \frac{\sin C}{46} = \frac{\sin 57°}{40.69}$

$\therefore \sin C = \frac{46 \sin 57°}{40.69}$

$\therefore \hat{C} = 71.5°$

$\therefore \hat{A} = 180° - 57° - 71.5°$

$= 51.5°$

b) $\cos F = \frac{d^2 + e^2 - f^2}{2de}$ (cosine rule)

$= \frac{185^2 + 290^2 - 145^2}{2(185)(290)}$

$\therefore \hat{F} = \cos^{-1} \frac{185^2 + 290^2 - 145^2}{2(185)(290)}$

$= 24.9°$

$\frac{\sin D}{d} = \frac{\sin F}{f}$ (sine rule)

$\therefore \frac{\sin D}{185} = \frac{\sin 24.9°}{145}$

$\therefore \sin D = \frac{185 \sin 24.9°}{145}$

$\therefore \hat{D} = 32.5°$

$\therefore \hat{E} = 180° - 24.9° - 32.5°$ (sum of $\angle$s of a $\triangle$)

$= 122.6°$

c) $\cos P = \frac{q^2 + r^2 - p^2}{2qr}$ (cosine rule)

$= \frac{8.8^2 + 9.2^2 - 5.4^2}{2(8.8)(9.2)}$

$\therefore \hat{P} = \cos^{-1} \frac{8.8^2 + 9.2^2 - 5.4^2}{2(8.8)(9.2)}$

$= 34.8°$

$\frac{\sin Q}{q} = \frac{\sin P}{p}$ (sine rule)

$\therefore \frac{\sin Q}{8.8} = \frac{\sin 34.8°}{5.4}$

$\therefore \sin Q = \frac{8.8 \sin 34.8°}{5.4}$

$\therefore \hat{Q} = 68.4°$

$\therefore \hat{R} = 180° - 34.8° - 68.4°$ (sum of $\angle$s of a $\triangle$)

$= 76.8°$

d) $x^2 = y^2 + z^2 - 2yz \cos X$ (cosine rule)

$\therefore YZ^2 = 33^2 + 49^2 - 2(33)(49) \cos 101°$

$\therefore YX = \sqrt{33^2 + 49^2 - 2(33)(49) \cos 101°}$

$= 64.09$ mm

$\frac{\sin Y}{y} = \frac{\sin X}{x}$ (sine rule)

$\therefore \frac{\sin Y}{33} = \frac{\sin 101°}{64.09}$

$\therefore \sin Y = \frac{33 \sin 101°}{64.09}$

$\therefore \hat{Y} = 30.4°$

$\therefore \hat{Z} = 180° - 101° - 30.4°$ (sum of $\angle$s of a $\triangle$)

$= 48.6°$

2. **a)** $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ (cosine rule)

$= \frac{6^2 + 7^2 - 5^2}{2(6)(7)}$

$\therefore \hat{C} = \cos^{-1} \frac{6^2 + 7^2 - 5^2}{2(6)(7)}$

$= 44.4°$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin A}{6} = \frac{\sin 44.4°}{5}$$

$$\therefore \sin A = \frac{6 \sin 44.4°}{5}$$

$$\therefore \hat{A} = 57.1°$$

$$\therefore \hat{B} = 180° - 44.4° - 57.1° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 78.5°$$

b) $$d^2 = e^2 + f^2 - 2ef \cos D \qquad \text{(cosine rule)}$$

$$\therefore EF^2 = 1.3^2 + 1.2^2 - 2(1.3)(1.2) \cos 124°$$

$$\therefore EF = \sqrt{1.3^2 + 1.2^2 - 2(1.3)(1.2) \cos 124°}$$

$$= 2.21 \text{ km}$$

$$\frac{\sin F}{f} = \frac{\sin D}{d} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin F}{1.2} = \frac{\sin 124°}{2.21}$$

$$\therefore \sin F = \frac{1.2 \sin 124°}{2.21}$$

$$\therefore \hat{F} = 26.8°$$

$$\therefore \hat{E} = 180° - 124° - 26.8° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 29.2°$$

c) $$j^2 = k^2 + l^2 - 2kl \cos J \qquad \text{(cosine rule)}$$

$$\therefore KL^2 = 137^2 + 169^2 - 2(137)(169) \cos 85°$$

$$\therefore KL = \sqrt{137^2 + 169^2 - 2(137)(169) \cos 85°}$$

$$= 208.07 \text{ units}$$

$$\frac{\sin K}{k} = \frac{\sin L}{l} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin K}{137} = \frac{\sin 85°}{208.07}$$

$$\therefore \sin K = \frac{137 \sin 85°}{208.07}$$

$$\therefore \hat{K} = 41.0°$$

$$\therefore \hat{L} = 180° - 85° - 41° \qquad \text{(sum of } \angle\text{s of a } \triangle)$$

$$= 54°$$

d) $$\cos M = \frac{n^2 + o^2 - m^2}{2no} \qquad \text{(cosine rule)}$$

$$= \frac{9^2 + 6^2 - 4^2}{2(9)(6)}$$

$$\therefore \hat{M} = \cos^{-1} \frac{9^2 + 6^2 - 4^2}{2(9)(6)}$$

$$= 20.7°$$

$$\frac{\sin O}{o} = \frac{\sin M}{m} \qquad \text{(sine rule)}$$

$$\therefore \frac{\sin O}{6} = \frac{\sin 20.7°}{4}$$

$$\therefore \sin O = \frac{6 \sin 20.7°}{4}$$

$$\therefore \hat{O} = 32.0°$$

$\therefore \hat{N} = 180° - 20.7° - 32°$ (sum of $\angle$s of a $\triangle$)
$= 127.3°$

e) $\cos R = \frac{p^2 + q^2 - r^2}{2pq}$ (cosine rule)

$= \frac{338^2 + 473^2 - 301^2}{2(338)(473)}$

$\therefore \hat{R} = \cos^{-1} \frac{338^2 + 473^2 - 301^2}{2(338)(473)}$

$= 39.3°$

$\frac{\sin P}{p} = \frac{\sin R}{r}$ (sine rule)

$\therefore \frac{\sin P}{338} = \frac{\sin 39.3°}{301}$

$\therefore \sin P = \frac{338 \sin 39.3°}{301}$

$\therefore \hat{P} = 45.3°$

$\therefore \hat{Q} = 180° - 39.3° - 45.3°$ (sum of $\angle$s of a $\triangle$)
$= 95.4°$

f) $y^2 = x^2 + z^2 - 2xz \cos Y$ (cosine rule)

$\therefore XZ^2 = 14^2 + 19.5^2 - 2(14)(19.5) \cos 21.6°$

$\therefore XZ = \sqrt{14^2 + 19.5^2 - 2(14)(19.5) \cos 21.6°}$

$= 8.28$ km

$\frac{\sin X}{x} = \frac{\sin Y}{y}$ (sine rule)

$\therefore \frac{\sin X}{14} = \frac{\sin 21.6°}{8.28}$

$\therefore \sin X = \frac{14 \sin 21.6°}{8.28}$

$\therefore \hat{X} = 38.5°$

$\therefore \hat{Z} = 180° - 21.6° - 38.5°$ (sum of $\angle$s of a $\triangle$)
$= 119.9°$

Exercise 14.4

1. $\frac{AB}{\sin C} = \frac{AC}{\sin B}$ (sine rule)

$\therefore \frac{AB}{\sin 49°} = \frac{45}{\sin 60°}$

$\therefore AB = \frac{45 \sin 49°}{\sin 60°}$

$= 29.22$ m

2. $x^2 = y^2 + z^2 - 2yz \cos X$ (cosine rule)

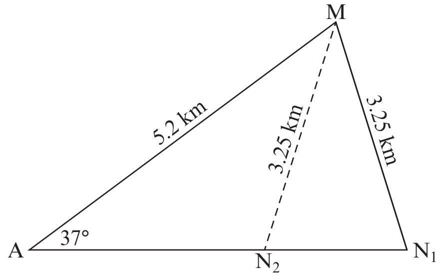
$\therefore YZ^2 = 95^2 + 80^2 - 2(95)(80) \cos 146°$

$\therefore YZ = \sqrt{95^2 + 80^2 - 2(95)(80) \cos 146°}$

$= 167$ km

3. **a)** The given angle, $\hat{A}$, is the non-included angle, and it is opposite the shorter of the two given sides (MN), $\therefore$ this could be an ambiguous case.

b)



c) **In $\triangle AMN_1$:**

$\frac{\sin N}{n} = \frac{\sin A}{a}$ (sine rule)

$\therefore \frac{\sin N}{5.2} = \frac{\sin 37°}{3.25}$

$\therefore \sin N = \frac{5.2 \sin 37°}{3.25}$

$\therefore \hat{N} = 74.3°$ (Use the acute $\angle$ for this $\triangle$.)

$\therefore \hat{M} = 180° - 37° - 74.3°$ (sum of $\angle$s of a $\triangle$)

$= 68.7°$

$\frac{m}{\sin M} = \frac{a}{\sin A}$ (sine rule)

$\therefore \frac{AN}{\sin 68.7°} = \frac{3.25}{\sin 37°}$

$\therefore AN = \frac{3.25 \sin 68.7°}{\sin 37°}$

$= 5.03$ km

In $\triangle AMN_2$:

$\hat{N} = 180° - 74.3°$ (Use the obtuse $\angle$ for this $\triangle$.)

$= 105.7°$

$\therefore \hat{M} = 180° - 37° - 105.7°$ (sum of $\angle$s of a $\triangle$)

$= 37.3°$

$\frac{m}{\sin M} = \frac{a}{\sin A}$ (sine rule)

$\therefore \frac{AN}{\sin 37.3°} = \frac{3.25}{\sin 37°}$

$\therefore AN = \frac{3.25 \sin 37.3°}{\sin 37°}$

$= 3.27$ km

$\therefore AN = 3.27$ km or 5.03 km

4. In △PQS:

$\cos P = \frac{q^2 + s^2 - p^2}{2qs}$ (cosine rule)

$= \frac{30^2 + 90^2 - 105^2}{2(30)(90)}$

$\therefore \hat{P} = \cos^{-1} \frac{30^2 + 90^2 - 105^2}{2(30)(90)}$

$= 112.0°$

$\therefore \hat{P}SR = 180° - 112.0°$ (coint. ∠s, PQ ∥ RS)

$= 68.0°$

In △PSR:

$s^2 = p^2 + r^2 - 2pr \cos S$ (cosine rule)

$\therefore PR^2 = 30^2 + 90^2 - 2(30)(90) \cos 68°$

$\therefore PR = \sqrt{30^2 + 90^2 - 2(30)(90) \cos 68°}$

$= 84$ mm

Assess your progress

1. a) $\hat{B} = 180° - 40° - 25°$ (sum of ∠s of a △)

$= 115°$

$\frac{a}{\sin A} = \frac{b}{\sin B}$ (sine rule)

$\therefore \frac{BC}{\sin 40°} = \frac{1.32}{\sin 115°}$

$\therefore BC = \frac{1.32 \sin 40°}{\sin 115°}$

$= 0.94$ km

$\frac{c}{\sin C} = \frac{b}{\sin B}$ (sine rule)

$\therefore \frac{AB}{\sin 25°} = \frac{1.32}{\sin 115°}$

$\therefore AB = \frac{1.32 \sin 25°}{\sin 115°}$

$= 0.62$ km

b) PR = PQ = 33 mm (given)

$\cos Q = \frac{p^2 + r^2 - q^2}{2pr}$ (cosine rule)

$= \frac{42^2 + 33^2 - 33^2}{2(42)(33)}$

$\therefore \hat{Q} = 50.5°$

$\therefore \hat{R} = 50.5°$ (isosceles △)

$\therefore \hat{P} = 180° - 50.5° - 50.5°$ (sum of ∠s of a △)

$= 79°$

c) $z^2 = x^2 + y^2 - 2xy \cos Z$ (cosine rule)

$\therefore XY^2 = 5.5^2 + 3^2 - 2(5.5)(3) \cos 63°$

$\therefore XY = \sqrt{5.5^2 + 3^2 - 2(5.5)(3) \cos 63°}$

$= 4.93$ m

$\frac{\sin Y}{y} = \frac{\sin Z}{z}$ (sine rule)

$\therefore \frac{\sin Y}{3} = \frac{\sin 63°}{4.93}$

$\therefore \sin Y = \frac{3 \sin 63°}{4.93}$

$\therefore \hat{Y} = 32.8°$

$\therefore \hat{X} = 180° - 63° - 32.8°$ (sum of ∠s of a △)

$= 84.2°$

d) $\frac{\sin N}{n} = \frac{\sin L}{l}$ (sine rule)

$\therefore \frac{\sin N}{18.6} = \frac{\sin 55°}{15.6}$

$\therefore \sin N = \frac{18.6 \sin 55°}{15.6}$

$\therefore \hat{N} = 77.6°$

$\hat{M} = 180° - 55° - 77.6°$ (sum of ∠s of a △)

$= 47.4°$

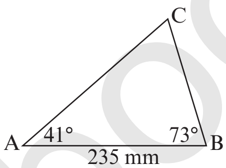
$\frac{m}{\sin M} = \frac{l}{\sin L}$ (sine rule)

$\therefore \frac{LN}{\sin 47.4°} = \frac{15.6}{\sin 55°}$

$\therefore LN = \frac{15.6 \sin 47.4°}{\sin 55°}$

$= 14.02$ cm

2. a)



$\hat{C} = 180° - 41° - 73°$ (sum of ∠s of a △)

$= 66°$

$\frac{b}{\sin B} = \frac{c}{\sin C}$ (sine rule)

$\therefore \frac{b}{\sin 73°} = \frac{235}{\sin 66°}$

$\therefore AC = \frac{235 \sin 73°}{\sin 66°}$

$= 246.00$ mm

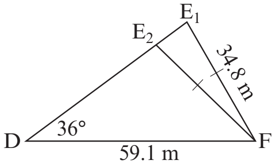
$\frac{a}{\sin A} = \frac{c}{\sin C}$ (sine rule)

$\therefore \frac{a}{\sin 41°} = \frac{235}{\sin 66°}$

$\therefore BC = \frac{235 \sin 41°}{\sin 66°}$

$= 168.76$ mm

b)



$\frac{\sin E}{e} = \frac{\sin D}{d}$ (sine rule)

$\therefore \frac{\sin E}{59.1} = \frac{\sin 36°}{34.8}$

$\therefore \sin E = \frac{59.1 \sin 36°}{34.8}$

$\therefore \hat{E} = 86.6°$ OR $\hat{E} = 180° - 86.6°$

$= 93.4°$

First solution, using Ê = 86.6°:

$\hat{F} = 180° - 36° - 86.6°$ (sum of ∠s of a △)

$= 57.4°$

$\frac{f}{\sin F} = \frac{d}{\sin D}$ (sine rule)

$\therefore \frac{DE}{\sin 57.4°} = \frac{34.8}{\sin 36°}$

$\therefore DE = \frac{34.8 \sin 57.4°}{\sin 36°}$

$= 49.88$ m

Second solution, using Ê = 93.4°:

$\hat{F} = 180° - 36° - 93.4°$ (sum of ∠s of a △)

$= 50.6°$

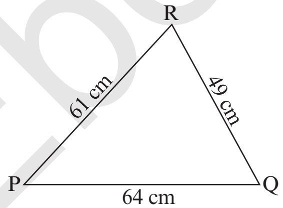
$\frac{f}{\sin F} = \frac{d}{\sin D}$ (sine rule)

$\therefore \frac{DE}{\sin 50.6°} = \frac{34.8}{\sin 36°}$

$\therefore DE = \frac{34.8 \sin 50.6°}{\sin 36°}$

$= 45.75$ m

c)



$\cos P = \frac{r^2 + q^2 - p^2}{2rq}$ (cosine rule)

$= \frac{64^2 + 61^2 - 49^2}{2(64)(61)}$

$\therefore \hat{P} = 46.1°$

$\frac{\sin Q}{q} = \frac{\sin P}{p}$ (sine rule)

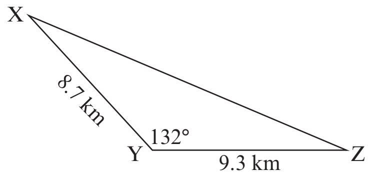
$\therefore \frac{\sin Q}{61} = \frac{\sin 46.1°}{49}$

$\therefore \sin Q = \frac{61 \sin 46.1°}{49}$

$\therefore \hat{Q} = 63.6°$

$\hat{R} = 180° - 46.1° - 63.6°$ (sum of ∠s of a △)

$= 70.3°$

d) 

$y^2 = x^2 + z^2 - 2xz \cos Y$ (cosine rule)

$\therefore XZ^2 = 9.3^2 + 8.7^2 - 2(9.3)(8.7) \cos 132°$

$\therefore XZ = \sqrt{9.3^2 + 8.7^2 - 2(9.3)(8.7) \cos 132°}$

$= 16.45$ cm

$\frac{\sin X}{x} = \frac{\sin Y}{y}$ (sine rule)

$\therefore \frac{\sin X}{9.3} = \frac{\sin 132°}{16.45}$

$\therefore \sin X = \frac{9.3 \sin 132°}{16.45}$

$\therefore \hat{X} = 24.8°$

$\therefore \hat{Z} = 180° - 132° - 24.8°$

$= 23.2°$

3. In $\triangle$PSR:

$\hat{P} = 180° - 111° - 30°$ (sum of $\angle$s of a $\triangle$)

$= 39°$

$\frac{r}{\sin R} = \frac{p}{\sin P}$ (sine rule)

$\therefore \frac{PS}{\sin 30°} = \frac{47}{\sin 39°}$

$\therefore PS = \frac{47 \sin 30°}{\sin 39°}$

$= 37.34$ km

$\frac{s}{\sin S} = \frac{p}{\sin P}$ (sine rule)

$\therefore \frac{PR}{\sin 111°} = \frac{47}{\sin 39°}$

$\therefore PR = \frac{47 \sin 111°}{\sin 39°}$

$= 69.72$ km

In $\triangle$PQR:

$p^2 = q^2 + r^2 - 2qr \cos P$ (cosine rule)

$\therefore QR^2 = 69.72^2 + 56^2 - 2(69.72)(56) \cos 53.5°$

$\therefore QR = \sqrt{69.72^2 + 56^2 - 2(69.72)(56) \cos 53.5°}$

$= 57.90$ km

$\therefore$ The perimeter of the farm

$= 56 + 57.90 + 47 + 37.34$

$= 198$ km (to the nearest kilometre)

TOPIC 15 Bearings

Introduction

The main focus of this topic is on bearings, although angles of elevation and depression are covered as well.

In previous years your students have applied angles of elevation and depression to right-angled triangles only, using the three basic trigonometric ratios. They now build on this knowledge as they apply angles of elevation and depression to non-right-angled triangles by using the sine and the cosine rules.

They will extend their understanding of the cardinal points of the compass to include 16 cardinal points.

Thus far, students have worked mostly with the three-digit notation for bearings, but now they will use cardinal notation for bearings as well. They will also apply bearings to non-right-angled triangles by using the sine and the cosine rules.

To conclude the topic, students will solve practical problems involving bearings in a variety of real-life contexts, including contexts that relate to sport, tourism and geography.

Common difficulties in this topic

Students need to become comfortable with complex multi-step calculations involving the trigonometric functions on their calculators. They need to know how to recognise the degree mode on their own model of calculator. If their calculators are not in degree mode, all their trigonometric calculations will be incorrect. You will need to remediate this individually for every different model of calculator.

If students do not have a calculator, they need to practise their skills in using the tables at the back of the Student's Book.

Preparation

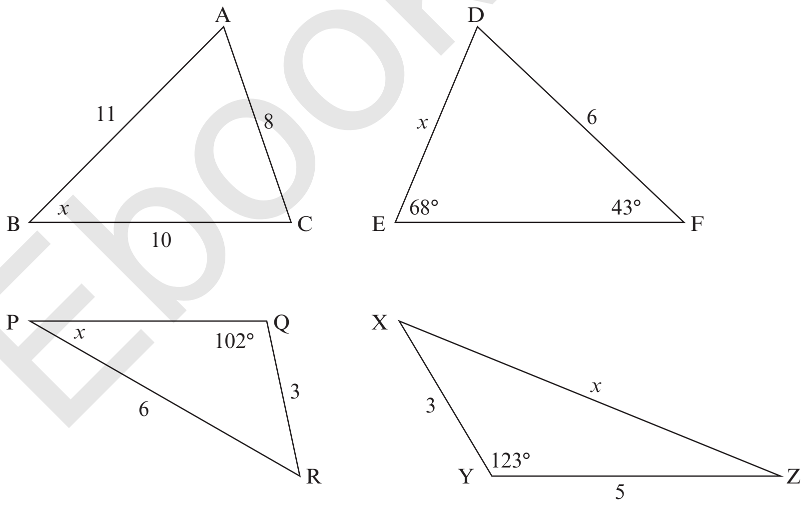
Prepare charts of the following and display them in your classroom for reference:

- angles of elevation and depression
- the 4, 8 and 16 cardinal points
- the sine and cosine rules.

Introduction for students

Explain to your class that they will be revising their existing knowledge of angles of elevation and depression, as well as bearings, and will then apply these concepts to triangles that are not right-angled. This means that they will be using the sine and the cosine rules to solve problems.

Draw these triangles on the board, and ask your class which rule (sine or cosine) they will use each time to solve for x, and why. If your students are not clear about these rules, revise them.



See if your students reply as follows:

- In △ABC they will use the cosine rule, because three sides are given.
- In △DEF they will use the sine rule, because two angles and a side are given.
- In △PQR they will use the sine rule, because two sides and the non-included angle are given.
- In △XYZ they will use the cosine rule, because two sides and the included angle are given.

Answers

Exercise 15.1

1. In $\triangle CFS$:

$\hat{S} = 64°$ (alt. $\angle$s)

$\frac{CF}{FS} = \tan \hat{S}$

$\therefore \frac{CF}{55} = \tan 64°$

$\therefore CF = 55 \times \tan 64°$

$= 113$ m

2. **a)** In $\triangle TWX$:

$\frac{TW}{WX} = \tan \hat{X}$

$\therefore \frac{TX}{200} = \tan 32.6°$

$\therefore TX = 200 \times \tan 32.6°$

$= 127.91$ m

b) In $\triangle TWY$:

$\frac{TW}{WY} = \tan Y$

$\therefore \frac{127.91}{WY} = \tan 43.5°$

$\therefore WY = \frac{127.91}{\tan 43.5°}$

$= 135$ m (to the nearest metre)

3. $\hat{B} = 68°$ (alt. $\angle$s, AB $\parallel$ CD)

$\therefore A\hat{C}B = 180° - 50° - 68°$ (sum of $\angle$s of a $\triangle$)

$= 62°$

$\therefore \frac{c}{\sin C} = \frac{a}{\sin A}$ (sine rule)

$\therefore \frac{AB}{\sin 62°} = \frac{53}{\sin 50°}$

$\therefore AB = \frac{53 \sin 62°}{\sin 50°}$

$= 61.1$ cm

4. **a)** $T\hat{G}K = (\beta - \alpha)$

b) In $\triangle GKT$:

$\cos G = \frac{k^2 + t^2 - b^2}{2kt}$ (cosine rule)

$\therefore \cos(\beta - \alpha) = \frac{(3x)^2 + (4x)^2 - (2x)^2}{2(3x)(4x)}$

$= \frac{9x^2 + 16x^2 - 4x^2}{24x^2}$

$= \frac{21x^2}{24x^2}$

$= \frac{7}{8}$

c) $\cos(\beta - \alpha) = \frac{7}{8}$

$\therefore \beta - \alpha = \cos^{-1} \frac{7}{8}$

$= 29°$ (to the nearest degree)

$\therefore 70° - \alpha = 29°$

$\therefore \alpha = 70° - 29°$

$= 41°$

Exercise 15.2

1. a) S b) NW c) ENE
   d) SSW e) SW f) WNW

2. a) NE b) E c) NNW
   d) W e) SW f) ESE

Exercise 15.3

1. a) 314°, N46°W b) 241°, S61°W

2. a) N45°E
   b) (i) 315°
   (ii) N45°W

3. a) 208°
   b) (i) 028°
   (ii) N28°E

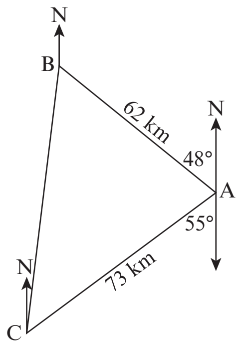
Exercise 15.4

1. a) obtuse $N\hat{K}L = 180° - 15°$ (coint. ∠s, North lines ∥)
   $= 165°$
   ∴ reflex $N\hat{K}L = 360° - 165°$ (∠s around a point)
   $= 195°$
   $= 180° + 15°$
   ∴ The bearing of L from K is S15°W.

   b) acute $N\hat{M}L = 180° - 96°$ (coint. ∠s, North lines ∥)
   $= 84°$
   ∴ reflex $N\hat{M}L = 360° - 84°$ (∠s around a point)
   $= 276°$
   $= 360° - 84°$
   ∴ The bearing of L from M is N84°W.

   c) $\cos M = \frac{k^2 + l^2 - m^2}{2kl}$ (cosine rule)
   $= \frac{122^2 + 283^2 - 270^2}{2(122)(283)}$
   ∴ $\hat{M} = \cos^{-1} \frac{122^2 + 283^2 - 270^2}{2(122)(283)}$
   $= 71.4°$
   ∴ acute $N\hat{M}K = 84° - 71.4°$
   $= 12.6°$
   ∴ reflex $N\hat{M}K = 360° - 12.6°$ (∠s around a point)
   $= 347.4°$
   $= 360° - 12.6°$
   ∴ The bearing of K from M is N12.6°W.

2. **a)**



b) $\text{A}\hat{\text{B}}\text{N} = 180° - 49°$ (coint. ∠s, North lines ∥)
$= 131°$
$\therefore$ The bearing from B to A is 131°.

c) $\text{N}\hat{\text{C}}\text{A} = 55°$ (alt. ∠s, North lines ∥)
$\therefore$ The bearing from C to A is 055°.

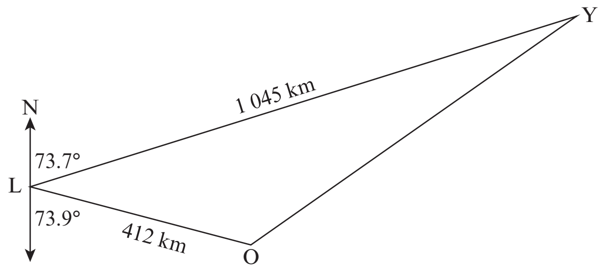
d) $\text{B}\hat{\text{A}}\text{C} = 180° - 49° - 55°$ (∠s on a straight line)
$= 76°$

e) $a^2 = b^2 + c^2 - 2bc \cos \text{A}$ (cosine rule)
$\therefore \text{BC}^2 = 73^2 + 62^2 - 2(73)(62) \cos 76°$
$\therefore \text{BC} = \sqrt{73^2 + 62^2 - 2(73)(62) \cos 76°}$
$= 83.57$ km

3. **a)** $\text{Q}\hat{\text{P}}\text{R} = 90° - 29°$
$= 61°$
$p^2 = q^2 + r^2 - 2qr \cos \text{P}$ (cosine rule)
$\therefore \text{QR}^2 = 520^2 + 350^2 - 2(520)(350) \cos 61°$
$\therefore \text{QR} = \sqrt{520^2 + 350^2 - 2(520)(350) \cos 61°}$
$= 465.22$ m

b) PL = 260 m
$p^2 = q^2 + f^2 - 2qf \cos \text{P}$ (cosine rule)
$\therefore \text{QL}^2 = 260^2 + 350^2 - 2(260)(350) \cos 61°$
$\therefore \text{QL} = \sqrt{260^2 + 350^2 - 2(260)(350) \cos 61°}$
$= 319.16$ m

4. **a)**



b) $Y\hat{L}O = 180° - 73.7° - 73.9°$ (∠s on a straight line)
$= 32.4°$

c) $l^2 = o^2 + y^2 - 2oy \cos L$ (cosine rule)
$\therefore OY^2 = 1\,045^2 + 412^2 - 2(1\,045)(412) \cos 32.4°$
$\therefore OY = \sqrt{1\,045^2 + 412^2 - 2(1\,045)(412) \cos 32.4°}$
$= 731$ km

Assess your progress

1. The ∠ of depression from Q to R is $P\hat{Q}R$.
$R\hat{P}Q = 26°$ (alt. ∠s, PQ ∥ RS)
In △PQR:
$\frac{\sin Q}{q} = \frac{\sin P}{p}$ (sine rule)
$\therefore \frac{\sin Q}{4.7} = \frac{\sin 26°}{2.4}$
$\therefore \sin Q = \frac{4.7 \sin 26°}{2.4}$
$\therefore \hat{Q} = \sin^{-1} \frac{4.7 \sin 26°}{2.4}$
$= 59.1°$

2. **a)** N72°E **b)** S57°E **c)** S88°W **d)** N15°W

3. **a)** 141° **b)** 342° **c)** 065° **d)** 231°

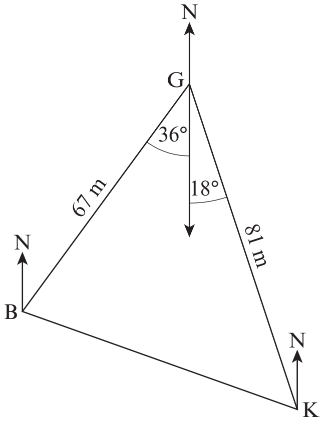
4. **a)** $Y\hat{X}Z = 115° - 57°$
$= 58°$
$\therefore \frac{\sin Y}{XZ} = \frac{\sin X}{YZ}$ (sine rule)
$\therefore \frac{\sin Y}{5} = \frac{\sin 58°}{4.5}$
$\therefore \sin Y = \frac{5 \sin 58°}{4.5}$
$\therefore \hat{Y} = \sin^{-1} \frac{5 \sin 58°}{4.5}$
$= 70.4°$
$\hat{Z} = 180° - 58° - 70.4°$ (sum of ∠s of a △)
$= 51.6°$
$\frac{XY}{\sin Z} = \frac{YZ}{\sin X}$ (sine rule)
$\therefore \frac{XY}{\sin 51.6°} = \frac{4.5}{\sin 58°}$
$\therefore XY = \frac{4.5 \sin 51.6°}{\sin 58°}$
$= 4.16$ m

b) obtuse $N\hat{Y}X = 180° - 57°$ (coint. ∠s, North lines ∥)
$= 123°$

$\therefore$ obtuse $N\hat{Y}Z = 360° - 123° - 70.4°$ (∠s around a point)
$= 166.6°$

$\therefore$ The bearing of Z from $y = 166.6°$.

5. a)



b) $B\hat{G}K = 18° + 36°$
$= 54°$

c) $g^2 = b^2 + k^2 - 2bk \cos G$ (cosine rule)

$\therefore BK^2 = 81^2 + 67^2 - 2(81)(67) \cos 54°$

$\therefore BK = \sqrt{81^2 + 67^2 - 2(81)(67) \cos 54°}$
$= 68$ m (to the nearest metre)

d) $\frac{\sin B}{b} = \frac{\sin G}{g}$ (sine rule)

$\therefore \frac{\sin B}{81} = \frac{\sin 54°}{68}$

$\therefore \sin B = \frac{81 \sin 54°}{68}$

$\therefore \hat{B} = \sin^{-1} \frac{81 \sin 54°}{68}$
$= 74.5°$

$N\hat{B}G = 36°$ (alt. ∠s, North lines ∥)

$\therefore N\hat{B}K = 36° + 74.5°$
$= 110.5°$
$= 180° - 69.5°$

$\therefore$ The bearing of K from B is S69.5°E.

TOPIC 16 Statistics

Introduction

In this topic, your students will build on their existing knowledge of statistics as they work with variance and standard deviation for the first time. They will also learn about cumulative frequencies and represent these in an ogive.

Common difficulties in this topic

The calculations in this topic are well within the scope of your students, but the notations that are used can be confusing. If your students struggle to calculate the variance and the standard deviation of data sets, refer them to the worked examples in the Student's Book. These serve as models for your students to follow.

When your students draw ogives, it is important that they work very neatly and accurately as they draw the scales on their axes and plot their points. They must also join their points with as smooth a curve as possible, making sure that their curve passes through all the points on their graph.

When estimating values from an ogive, your students should be very careful to estimate accurately, bearing on mind the scale on each axis. They should also realise that their answers are estimates only and that they might differ slightly from the model answers provided below. It is acceptable that their answers do not agree exactly with the model answers, but they should be reasonably close.

Preparation

Look for examples of statistics in daily life in sources like newspapers and magazines, particularly relating to the capital market. Make a collage of these on a chart and display the chart on the walls of your classroom. As you work through this topic with your students, ask them to look for more examples that they could bring to school and add to the collage.

Introduction for students

Write the following terms on the board:

mean	median	mode	range
ungrouped data	grouped data	frequency table	tally
histogram			

These are all terms that were dealt with in SS1. Ask a volunteer from your class to choose any term and then explain it to the class. Once the correct explanation has been given, cross the term out and ask a new volunteer to choose one of the remaining terms and explain it to the class. Continue until all the terms have been discussed.

Answers

Exercise 16.1

1. **a)** Mean = $\frac{3 + 5 + 9 + 12 + 17 + 21 + 21 + 26 + 33}{9}$ = 16.33 (correct to two decimal places)

b) There are 9 data values in the data set. Middle value is 5th data value, which is 17, ∴ median = 17.

c) Mode = 21.

d) Range = 33 − 3 = 30.

2. Ordered data set: {2, 15, 16, 21, 28, 33, 41, 41, 42, 46, 52, 52, 54, 59}.

a) Mean = $\frac{2 + 15 + 16 + 21 + 28 + 33 + 41 + 41 + 42 + 46 + 52 + 52 + 54 + 59}{14}$
= 35.86 (correct to two decimal places).

b) There are 14 data values in the data set. Middle value lies halfway between the 7th and the 8th data values, which are 41 and 41, ∴ median = 41.

c) Modes are 41 and 52.

d) Range = 59 − 2 = 57.

3. Ordered data set: {−7.7, −7.5, −5.9, −0.1, 0.3, 3.6, 3.9, 5, 5.5, 5.6, 8.4, 8.9, 9.6}.

a) Mean
= $\frac{-7.7 - 7.5 - 5.9 - 0.1 + 0.3 + 3.6 + 3.9 + 5 + 5.5 + 5.6 + 8.4 + 8.9 + 9.6}{13}$
= 2.28 (correct to two decimal places).

b) There are 13 data values in the data set. Middle value is 7th data value, which is 3.9, ∴ median = 3.9.

c) There is no mode.

d) Range = 9.6 − (−7.7) = 17.3.

Exercise 16.2

1. a)

Number of roses	Tally	Frequency
0	𝍷𝍷𝍷	3
1	𝍷𝍷𝍷𝍷	4
2	𝍸 𝍷	6
3	𝍸 𝍷	6
4	𝍷𝍷	2

b) (i) Sum of all the data values: $(0 \times 3) + (1 \times 4) + (2 \times 6) + (3 \times 6) + (4 \times 2) = 42$. There are 21 data values. Mean is $\frac{42}{21} = 2$ roses.

(ii) Middle value is 11th data value, which is 2, $\therefore$ the median is 2 roses.

(iii) Modes are 2 and 3 roses, because both of these have the highest frequency (6).

(iv) Range = 4 − 0 = 4 roses.

2. a)

Number of books	Tally	Frequency
0	𝍸	5
1	𝍸 𝍷	6
2	𝍸 𝍷𝍷𝍷	8
3	𝍸 𝍷𝍷	7
4	𝍸 𝍸 𝍷𝍷	12
5	𝍸	5
6	𝍷𝍷	2
7	𝍷	1
8	𝍷𝍷	2

b) (i) Sum of all the data values: $(0 \times 5) + (1 \times 6) + (2 \times 8) + (3 \times 7) + (4 \times 12) + (5 \times 5) + (6 \times 2) + (7 \times 1) + (8 \times 21) = 151$. There are 48 data values. Mean is $\frac{151}{48} = 3.15$ books (correct to two decimal places).

(ii) Middle value lies halfway between the 24th and the 25th data values, which are 3 and 3, $\therefore$ the median is 3 books.

(iii) Mode = 4 books, because this is the data value with the highest frequency (12).

(iv) Range = 8 − 0 = 8 books.

Exercise 16.3

1. $\bar{x} = \frac{1 + 4 + 7 + 12 + 20 + 33 + 49}{7} = 18$

Data value (x)	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)
1	18	–17
4	18	–14
7	18	–11
12	18	–6
20	18	2
33	18	15
49	18	31

2. **a)** $\bar{x} = \frac{6 + 0 + (-3) + 1 + 5 + 4 + (-4) + 3}{8} = 1.5$

b)

Data value (x)	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)
6	1.5	4.5
0	1.5	–1.5
–3	1.5	–4.5
1	1.5	–0.5
5	1.5	3.5
4	1.5	2.5
–4	1.5	–5.5
3	1.5	1.5

3. **a)** $\bar{x} = \frac{29 + 14 + 17 + 25 + 37 + 16 + 23 + 28 + 40 + 31}{10}$

b)

Number of baboons (x)	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)
29	26	3
14	26	–12
17	26	–9
25	26	–1
37	26	11
16	26	–10
23	26	–3
28	26	2
40	26	14
31	26	5

Exercise 16.4

1.

Data value (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
0	13	–13	169
3	13	–10	100
7	13	–6	36
11	13	–2	4
12	13	–1	1
21	13	8	64
23	13	10	100
27	13	14	196

$$\Sigma(x - \overline{x})^2 = 169 + 100 + 36 + 4 + 1 + 64 + 100 + 196$$
$$= 670$$
$$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{670}{8} = 83.75$$

2. **a)** $\overline{x} = \frac{2 + 4 + 6 + 8 + 10 + 12}{6} = 7$

b)

Data value (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
2	7	–5	25
4	7	–3	9
6	7	–1	1
8	7	1	1
10	7	3	9
12	7	5	25

$$\Sigma(x - \overline{x})^2 = 25 + 9 + 1 + 1 + 9 + 25 = 70$$
$$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{70}{6} = 11.67$$

3. **a)** $\overline{x} = \frac{-12 - 7 - 2 + 0 + 3 + 6 + 9 + 10 + 14 + 28}{10} = 4.9$

b)

Data value (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
–12	4.9	–16.9	285.61
–7	4.9	–11.9	141.61
–2	4.9	–6.9	47.61
0	4.9	–4.9	24.01
3	4.9	–1.9	3.61
6	4.9	1.1	1.21
9	4.9	4.1	16.81
10	4.9	5.1	26.01
14	4.9	9.1	82.81
28	4.9	23.1	533.61

$$\Sigma(x - \overline{x})^2 = 285.61 + 141.61 + 47.61 + 24.01 + 3.61 + 1.21 + 16.81 + 26.01 + 82.81 + 533.61 = 1\,162.9$$

$$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{1\,162.9}{12} = 96.91$$

4. $\overline{x} = \frac{2 + 4 + 9 + 0 + 6 + 3 + 2 + 1 + 5 + 0}{10} = 3.2$

Number of movies (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
2	3.2	–1.2	1.44
4	3.2	0.8	0.64
9	3.2	5.8	33.64
0	3.2	–3.2	10.24
6	3.2	2.8	7.84
3	3.2	–0.2	0.04
2	3.2	–1.2	1.44
1	3.2	–2.2	4.84
5	3.2	1.8	3.24
0	3.2	–3.2	10.24

$$\Sigma(x - \overline{x})^2 = 1.44 + 0.64 + 33.64 + 10.24 + 7.84 + 0.04 + 1.44 + 4.84 + 3.24 + 10.24 = 73.6$$

$$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{73.6}{10} = 7.36$$

Exercise 16.5

1. **a)** $\Sigma = \sqrt{1.44} = 1.2$ **b)** $\Sigma = \sqrt{32.75} = 5.72$
c) $\Sigma = \sqrt{192.68} = 13.88$ **d)** $\Sigma = \sqrt{4\,091.03} = 63.96$

2. **a)** $\Sigma^2 = 0.3^2 = 0.09$ **b)** $\Sigma^2 = 1.09^2 = 1.19$
c) $\Sigma^2 = 29.83^2 = 889.83$ **d)** $\Sigma^2 = 508.15^2 = 258\,216.42$

3. **a)** $\overline{x} = \frac{1 + 2 + 3 + 4 + 5}{5} = 3$

Data value (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
1	3	–2	4
2	3	–1	1
3	3	0	0
4	3	1	1
5	3	2	4

$\Sigma(x - \overline{x})^2 = 4 + 1 + 0 + 1 + 4 = 10$

$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{10}{5} = 2$

$\therefore \Sigma = \sqrt{2}$

b) $\overline{x} = \frac{14 + 15 + 16 + 17 + 18}{5} = 16$

Data value (x)	Mean $(\overline{x})$	Deviation $(x - \overline{x})$	Square of deviation $(x - \overline{x})^2$
14	16	–2	4
15	16	–1	1
16	16	0	0
17	16	1	1
18	16	2	4

$\Sigma(x - \overline{x})^2 = 4 + 1 + 0 + 1 + 4 = 10$

$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{10}{5} = 2$

$\therefore \Sigma = \sqrt{2}$

c) $\overline{x} = \frac{2 + 4 + 6 + 8 + 10}{5} = 6$

Data value (x)	Mean $(\overline{x})$	Deviation $(x - \overline{x})$	Square of deviation $(x - \overline{x})^2$
2	6	–4	16
4	6	–2	4
6	6	0	0
8	6	2	4
10	6	4	16

$\Sigma(x - \overline{x})^2 = 16 + 4 + 0 = 4 + 16 = 40$

$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{40}{5} = 8$

$\therefore \Sigma = \sqrt{8} = 2\sqrt{2}$

d) $\overline{x} = \frac{100 + 200 + 300 + 400 + 500}{5} = 300$

Data value (x)	Mean $(\overline{x})$	Deviation $(x - \overline{x})$	Square of deviation $(x - \overline{x})^2$
100	300	–200	40 000
200	300	–100	10 000
300	300	0	0
400	300	100	10 000
500	300	200	40 000

$$\Sigma(x - \bar{x})^2 = 40\,000 + 10\,000 + 0 + 10\,000 + 40\,000$$
$$= 100\,000$$
$$\therefore \Sigma^2 = \frac{\Sigma(x - \bar{x})^2}{n} = \frac{100\,000}{5} = 20\,000$$
$$\therefore \Sigma = \sqrt{20\,000} = 10\sqrt{2}$$

4. **a)** Both sets consist of five consecutive integers and both sets have the same standard deviation.
   b) The numbers in the second set are double those in the first set, and the standard deviation of the second set is double that of the first set.
   c) The numbers in the second set are one hundred times those in the first set, and the standard deviation of the second set is one hundred times that of the first set.

5. Yes. A data set will have a variance and a standard deviation of zero if all the numbers in the data set are identical.

6. No. The variance is the sum of squares and so it cannot ever be negative. The standard deviation is the square root of a non-negative number and so it cannot ever be negative.

Exercise 16.6

1.

Mass (g)	201–250	251–300	301–350	351–400
Lower class boundary (g)	200.5	250.5	300.5	350.5
Upper class boundary (g)	250.5	300.5	350.5	400.5
Midpoint (g)	225.5	275.5	325.5	375.5

2.

Length (m)	11–16	17–22	23–28	29–34	35–40	41–46
Lower class boundary (m)	10.5	16.5	22.5	28.5	34.5	40.5
Upper class boundary (m)	16.5	22.5	28.5	34.5	40.5	46.5
Midpoint (m)	13.5	19.5	25.5	31.5	37.5	43.5

3.

Time (h)	0–1.4	1.5–2.9	3–4.4	4.5–5.9	6–7.4	7.5–8.9
Lower class boundary (h)	–0.05	1.45	2.95	4.45	5.95	7.45
Upper class boundary (h)	1.45	2.95	4.45	5.95	7.45	8.95
Midpoint (h)	0.7	2.2	3.7	5.2	6.7	8.2

Exercise 16.7

1.

Temperature of water (°C)	Tally	Frequency
[–17, –13)	|	1
[–13, –9)	||	2
[–9, –5)	|||	3
[–5, –1)	𝍸	5
[–1, 3)	𝍸 ||||	9
[3, 7)	𝍸	5
[7, 11)	𝍸 ||	7

2.

Number of meals	Tally	Frequency
91–100	𝍸 ||||	9
101–110	𝍸 |||	8
111–120	|||	3
121–130	𝍸 𝍸 ||	12
131–140	𝍸 |||	8

3.

Masses of papayas (g)	Tally	Frequency
700–799	𝍸 |	6
800–899	𝍸 ||	7
900–999	𝍸 𝍸 |	11
1 000–1 099	𝍸 ||||	9
1 100–1 199	𝍸 𝍸 ||||	14
1 200–1 299	𝍸 ||||	9

Exercise 16.8

1. a) (i) First calculate the midpoints of the classes.

Distance (km)	(0, 2]	(2, 4]	(4, 6]	(6, 8]	(8, 10]
Midpoint	1	3	5	7	9
Frequency	29	34	16	11	5

Estimated sum of all the data values: $(1 \times 29) + (3 \times 34) + (5 \times 16) + (7 \times 11) + (9 \times 5) = 333$. There are $(29 + 34 + 16 + 11 + 5) = 95$ data values. The estimated mean $= \frac{333}{95} = 3.51$ km (correct to two decimal places).

(ii) There are 95 data values. The median is the 48th data value. The 48th data value is the 19th data value in the class (2, 4] (because 48 − 29 = 19). The estimated median = 2 + $\frac{19}{34}$ × 2 = 2.24 km (correct to two decimal places).

b) Modal class is (2, 4], because it has the highest frequency.

c) Largest possible range = 10 − 0 = 10 km. Smallest possible range = 8 − 2 = 6 km, ∴ the range lies between 6 and 10 km.

2. **a)** **(i)** First calculate the midpoints of the classes.

Speed (km/h)	(80, 90]	(90, 100]	(100, 110]	(110, 120]
Midpoint	85	95	105	115
Frequency	57	46	21	8

Estimated sum of all the data values: (85 × 57) + (95 × 46) + (105 × 21) + (115 × 8) = 12 340. There are (57 + 46 + 21 + 8) = 132 data values. The estimated mean = $\frac{12\,340}{132}$ = 93.48 km/h (correct to two decimal places).

(ii) There are 132 data values. The median lies halfway between the 66th and the 67th data values. The 66.5th data value is the 9.5th data value in the class (90, 100] (because 66.5 − 57 = 9.5). The estimated median = 90 + $\frac{9.5}{46}$ ×10 = 92.05 km/h (correct to two decimal places).

b) Modal class is (80, 90], because it has the highest frequency.

c) Largest possible range = 120 − 80 = 40 km/h. Smallest possible range = 110 − 90 = 20 km/h, ∴ the range lies between 20 and 40 km/h.

Exercise 16.9

1.

Time (min)	Midpoint (x)	Frequency	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)
(0, 10]	5	4	32.5	–27.5
(10, 20]	15	9	32.5	–17.5
(20, 30]	25	18	32.5	–7.5
(30, 40]	35	23	32.5	2.5
(40, 50]	45	15	32.5	12.5
(50, 60]	55	7	32.5	22.5

2. **a)** Midpoints are: −25, −15, −5, 5, 15, 25, 35.

b) Estimated sum of all the data values: (−25 × 8) + (−15 × 9) + (−5 × 9) + (5 × 15) + (15 × 9) + (25 × 6) + (35 × 4) = 120. There are (8 + 9 + 9 + 15 + 9 + 6 + 4) = 60 data values. $\bar{x} = \frac{120}{60} = 2$ °C.

c)

Temperature (°C)	Midpoint (x)	Frequency	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)
(–30, –20]	–25	8	2	–27
(–20, –10]	–15	9	2	–17
(–10, 0]	–5	9	2	–7
(0, 10]	5	15	2	3
(10, 20]	15	9	2	13
(20, 30]	25	6	2	23
(30, 40]	35	4	2	33

3. **a)** First calculate the midpoints of the classes.

Number of tickets sold	1–20	21–40	41–60	61–80	81–100
Midpoint	10.5	30.5	50.5	70.5	90.5
Number of students	61	53	36	30	20

Estimated sum of all the data values: (10.5 × 61) + (30.5 × 53) + (50.5 × 36) + (70.5 × 30) + (90.5 × 20) = 8 000. There are (61 + 53 + 36 + 30 + 20) = 200 data values. $\bar{x} = \frac{8\,000}{200} = 40$ tickets.

b)

Number of tickets sold	Midpoint (x)	Frequency	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)
1–20	10.5	61	40	–29.5
21–40	30.5	53	40	–9.5
41–60	50.5	36	40	10.5
61–80	70.5	30	40	30.5
81–100	90.5	20	40	50.5

Exercise 16.10

1.

Length (m)	Midpoint (x)	Frequency	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)	Square of deviation $(x - \bar{x})^2$
(0, 4]	2	5	14.2	–12.2	148.84
(4, 8]	6	7	14.2	–8.2	67.24
(8, 12]	10	10	14.2	–4.2	17.64
(12, 16]	14	13	14.2	–0.2	0.04
(16, 20]	18	12	14.2	3.8	14.44
(20, 24]	22	9	14.2	7.8	60.84
(24, 28]	26	4	14.2	11.8	139.24

a) $\Sigma[(x - \bar{x})^2 \times f] = (148.84 \times 5) + (67.24 \times 7)$ $+ (17.64 \times 10) + (0.04 \times 13) + (14.44 \times 12)$ $+ (60.84 \times 9) + (139.24 \times 4) = 2\,669.6$

Total number of data values
$= (5 + 7 + 10 + 13 + 12 + 9 + 4)$
$= 60$

$\therefore \Sigma^2 = \Sigma\frac{(x - \bar{x})^2 \times f}{n}$

$= \frac{2\,669.6}{60}$

$= 44.49$ (correct to two decimal places)

b) $\Sigma = \sqrt{\Sigma^2}$

$= \sqrt{44.49}$

$= 6.67$ (correct to two decimal places)

2. a) First calculate the midpoints of the classes.

Mass (tonnes)	1–1.4	1.5–1.9	2–2.4	2.5–2.9	3–3.4	3.5–3.9
Midpoint	1.2	1.7	2.2	2.7	3.2	3.7
Frequency	10	16	31	17	27	19

Estimated sum of all the data values: (1.2×10) $+ (1.7 \times 16) + (2.2 \times 31) + (2.7 \times 17) + (3.2 \times 27)$ $+ (3.7 \times 19) = 310$. There are $(10 + 16 + 31 + 17$ $+ 27 + 19) = 120$ data values.

$\bar{x} = \frac{310}{120} = 2.58$ tonnes (correct to two decimal places).

b)

Mass (tonnes)	Midpoint (x)	Frequency	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)	Square of deviation $(x - \bar{x})^2$
1–1.4	1.2	10	2.58	–1.38	1.9044
1.5–1.9	1.7	16	2.58	–0.88	0.7744
2–2.4	2.2	31	2.58	–0.38	0.1444
2.5–2.9	2.7	17	2.58	0.12	0.0144
3–3.4	3.2	27	2.58	0.62	0.3844
3.5–3.9	3.7	19	2.58	1.12	1.2544

(i) $\Sigma[(x - \bar{x})^2 \times f] = (1.9044 \times 10) + (0.7744 \times 16) + (0.1444 \times 31) + (0.0144 \times 17) + (0.3844 \times 27) + (1.2544 \times 19) = 70.368$

$\therefore \Sigma^2 = \Sigma\frac{(x - \bar{x})^2 \times f}{n}$

$= \frac{70.368}{120}$

$= 0.59$ (correct to two decimal places)

(ii) $\Sigma = \sqrt{\Sigma^2}$

$= \sqrt{0.59}$

$= 0.77$ (correct to two decimal places)

3. a) First calculate the midpoints of the classes.

Area (cm^2)	301–350	351–400	401–450	451–500	501–550
Midpoint	325.5	375.5	425.5	475.5	525.5
Frequency	5	8	15	12	10

Estimated sum of all the data values: (325.5 × 5) + (375.5 × 8) + (425.5 × 15) + (475.5 × 12) + (525.5 × 10) = 21 975. There are (5 + 8 + 15 + 12 + 10) = 50 data values. $\bar{x} = \frac{21\,975}{50} = 439.5$ cm^2

b)

Area (cm^2)	Midpoint (x)	Frequency	Mean ($\bar{x}$)	Deviation ($x - \bar{x}$)	Square of deviation $(x - \bar{x})^2$
301–350	325.5	5	439.5	–114	12 996
351–400	375.5	8	439.5	–64	4 096
401–450	425.5	15	439.5	–14	196
451–500	475.5	12	439.5	36	1 296
501–550	525.5	10	439.5	86	7 396

(i) $\Sigma[(x - \overline{x})^2 \times f] = (12\,996 \times 5) + (4\,096 \times 8)$
$+ (196 \times 15) + (1\,296 \times 12) + (7\,396 \times 20)$
$= 190\,200$

$\therefore \Sigma^2 = \Sigma\frac{(x - \overline{x})^2 \times f}{n}$

$= \frac{190\,200}{50}$

$= 3\,804$

(ii) $\Sigma = \sqrt{\Sigma^2}$
$= \sqrt{3\,804}$
$= 61.68$ (correct to two decimal places)

Exercise 16.11

1. **a)** $\overline{x} - \Sigma = 22.6 - 6.45 = 16.15$
$\overline{x} + \Sigma = 22.6 + 6.45 = 29.05$
$\therefore$ 35 marks lie within one standard deviation from the mean.

b) The marks that lie outside of one standard deviation from the mean are 2, 9, 10, 16, 30, 30, 30, 30, 31 and 38.

c) 40% of 40 = 16, so three students failed the test.

d) Students' own opinions and reasons. Accept any reasonable opinion, as long as it is logical and is explained clearly. Below is a suggested answer.
The test was not too difficult, because out of 45 students, only three failed. Furthermore, the marks of the three students who failed were more than one standard deviation below the mean. This means that they did particularly badly in comparison with the rest of the class, and most likely deserved to fail.

2. **a)** $\overline{x} = \frac{500 + 100 + 200 + 800 + 200 + 900}{6} = 450$

Shift	Voters (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
06:00–08:00	500	450	50	2 500
08:00–10:00	100	450	–350	122 500
10:00–12:00	200	450	–250	62 500
12:00–14:00	800	450	350	122 500
14:00–16:00	200	450	–250	62 500
16:00–18:00	900	450	450	202 500

$\Sigma(x - \overline{x})^2 = 2\ 500 + 122\ 500 + 62\ 500 + 122\ 500$
$+ 62\ 500 + 202\ 500 = 575\ 000$

$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{575\ 000}{6} = 95\ 833$

b) $\Sigma = \sqrt{95\ 833} = 310$

c) $\overline{x} - \Sigma = 450 - 310 = 140$
$\overline{x} + \Sigma = 450 + 310 = 760$
$\therefore$ The 08:00–10:00 shift, the 12:00–14:00 shift and the 16:00–18:00 shift do not fall within one standard deviation from the mean.

d) The data does suggest that the by-election took place on a normal working day. 500 voters voted before work. The other two peak times were during the lunch break (800 voters) and after work (900 voters).

3. **a)** $\overline{x} = \frac{5 + 10 + 12 + 18 + 21 + 28 + 40 + 54}{8} = 23.5$

Mass (x)	Mean $(\overline{x})$	Deviation $(x - \overline{x})$	Square of deviation $(x - \overline{x})^2$
5	23.5	–18.5	342.25
10	23.5	–13.5	182.25
12	23.5	–11.5	132.25
18	23.5	–5.5	30.25
21	23.5	–2.5	6.25
28	23.5	4.5	20.25
40	23.5	16.5	272.25
54	23.5	30.5	930.25

$\Sigma(x - \overline{x})^2 = 342.25 + 182.25 + 132.25 + 30.25$
$+ 6.25 + 20.25 + 272.25 + 930.25 = 1\ 916$

$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{1\ 916}{8} = 239.5$

$\therefore \Sigma = \sqrt{239.5} = 15.48$ kg

b) $\overline{x} - \Sigma = 23.5 - 15.48 = 8.02$ kg
$\overline{x} + \Sigma = 23.5 + 15.48 = 38.98$ kg
$\therefore$ Five of these masses fall within one standard deviation from the mean.

4. **a)** All the measurements were identical.

b) Students' own explanations. Accept any reasonable explanation, as long as it is logical and is explained

clearly. Below is one possible explanation.
The sum of the five measurements is 72° × 5 = 360°. If all the students had measured the angle correctly, the sum would have been 60° × 5 = 300°. Assuming that only one student made a mistake, his or her measurement would have been 120°. This is possible if the student had read the measurement off the protractor incorrectly, i.e. from the wrong side.

Exercise 16.12

1. a) No. Numbers of things are discrete data, because they are counted, not measured.

b) Yes. Temperatures are continuous data, because temperatures are measured, not counted.

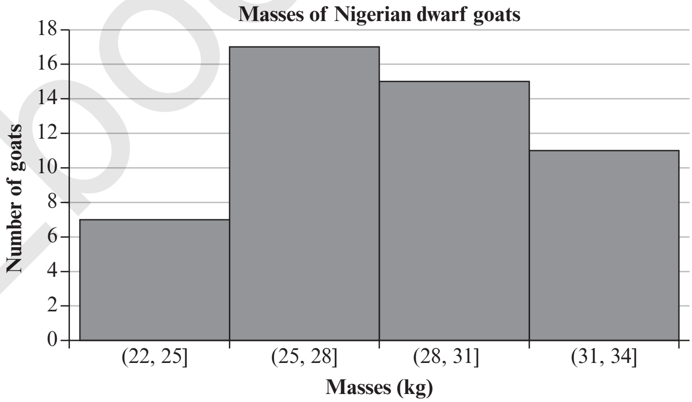
c) No. Opinions are not continuous data.

d) No. Marks are discrete data because marks are counted, not measured.

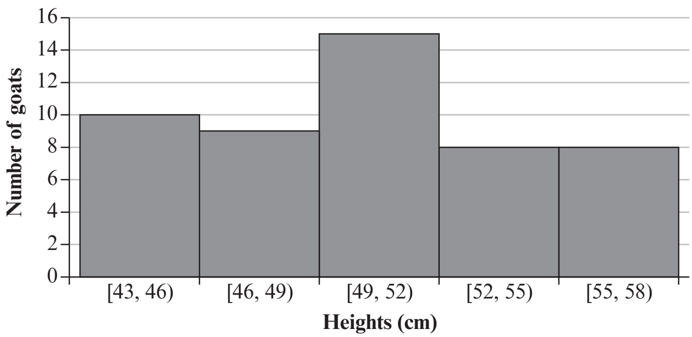
e) Yes. Heights are continuous data, because heights are measured, not counted.

2. a) 50 goats are represented in each table.

b)



c)



Exercise 16.13

1.

Ages of the population of a small village (years)	Frequency	Cumulative frequency
0–10	73	73
11–20	86	159
21–30	65	224
31–40	81	305
41–50	72	377
51–60	58	435
61–70	37	472
71–80	19	491
81–90	7	498
91–100	2	500

2.

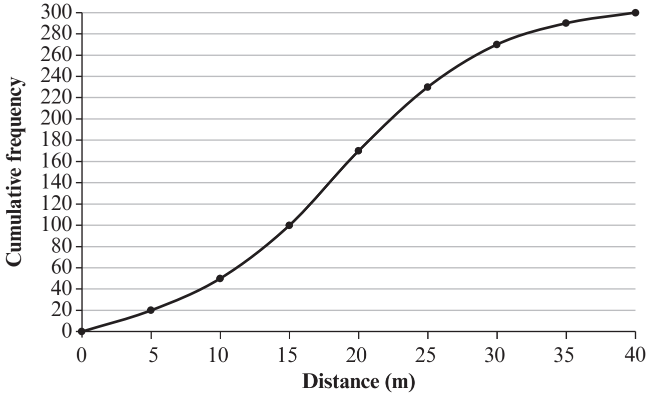
Ages of the population of a small village (years)	Frequency	Cumulative frequency
0–20	159	159
21–40	146	305
41–60	130	435
61–80	56	491
81–100	9	500

Exercise 16.14

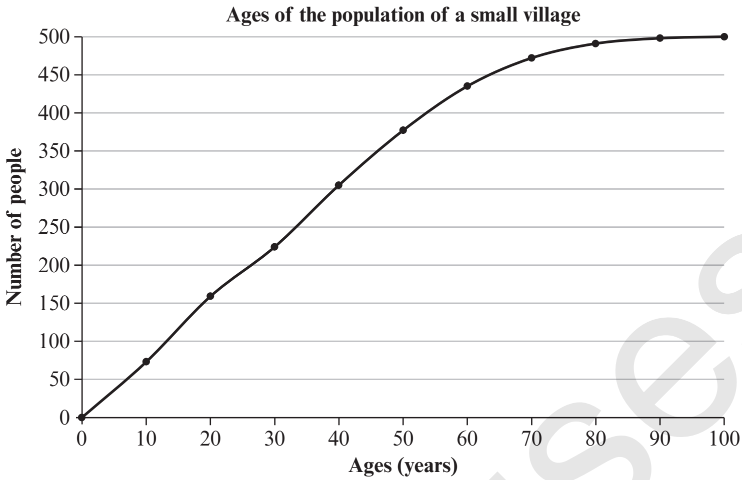
1. a)

Distance (m)	(0, 5]	(5, 10]	(10, 15]	(15, 20]	(20, 25]	(25, 30]	(30, 35]	(35, 40]
Frequency	20	30	50	70	60	40	20	10
Cumulative frequency	20	50	100	170	230	270	290	300

b)

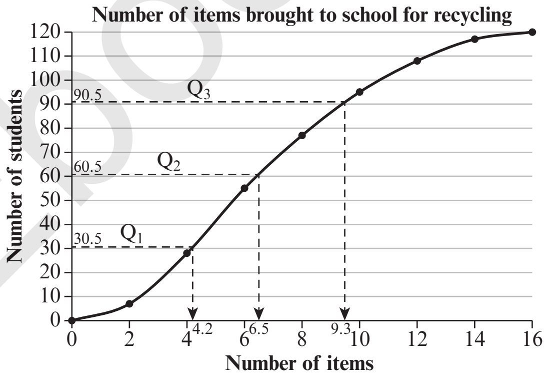


2.



Exercise 16.15

1. **a)** 120 students took part in this recycling effort.
   b) The maximum number of items that any one student brought to school was 16.
   c) The solutions to questions (i)–(iii) are shown in the ogive below and are discussed in the answers that follow.



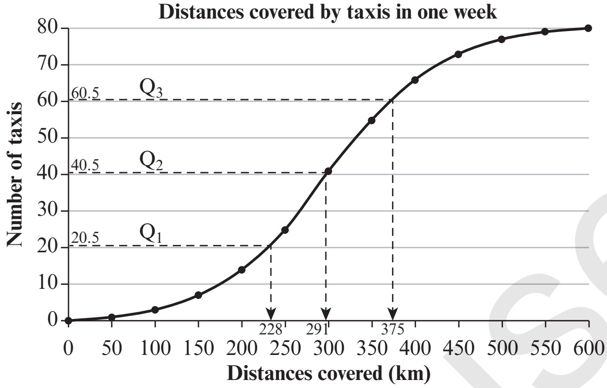
(i) There are 120 data values. Q_2 lies between the 60th and the 61st data values (i.e. 60.5). The estimated value of Q_2 is 6.5.

(ii) Q_1 lies between the 30th and the 31st data values (i.e. 30.5). The estimated value of Q_1 is 4.2.

(iii) Q_3 lies between the 90th and the 91st data values (i.e. 90.5). The estimated value of Q_3 is 9.3.

d) Without having to draw a dotted line, it is clear that a student who brought eight items to school for recycling will fall between Q_2 and Q_3.

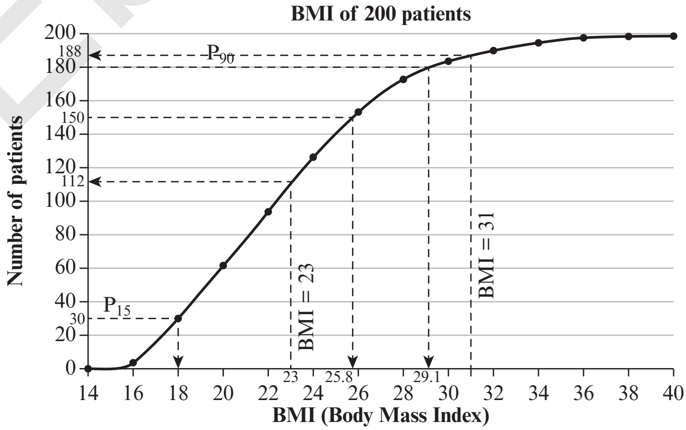
2. **a)** 80 taxis operated during that week.
b) The maximum distance that any one taxi covered was 600 km.



c) **(i)** There are 80 data values. Q_2 lies between the 40th and the 41st data values (i.e. 40.5). The estimated value of Q_2 is 291 km.
(ii) Q_1 lies between the 20th and the 21st data values (i.e. 20.5). The estimated value of Q_1 is 228 km.
(iii) Q_3 lies between the 60th and the 61st data values (i.e. 60.5). The estimated value of Q_3 is 375.
d) The value of 291 km for Q_2 means that half of the taxis covered a distance of 291 km or less, and the other half of the taxis covered a distance of 291 km or more.

Exercise 16.16

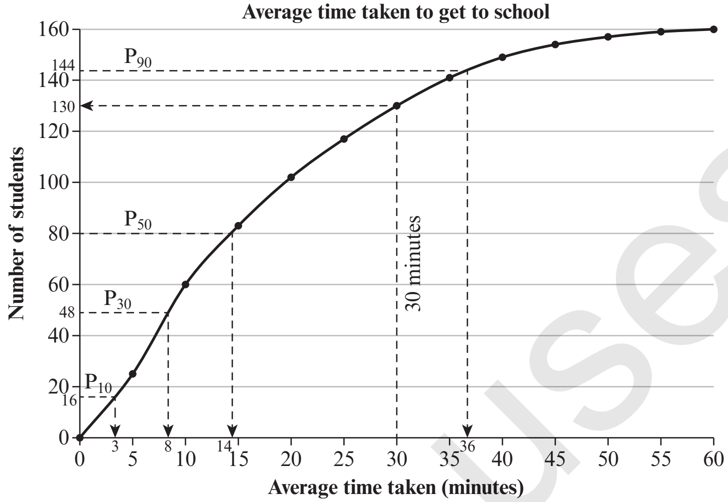
1.



a) P_{75} corresponds to 75% of the total frequency.
$\frac{75}{100} \times 200 = 150.$
The estimated value of P_{75} is a BMI of 25.8.

b) A BMI of 31 on the horizontal axis corresponds to the value of 188 on the vertical axis.

2.



a) (i) P_{10} corresponds to 10% of the total frequency.

$\frac{10}{100} \times 160 = 16$.

The estimated value of P_{10} is 3 minutes (to the nearest minute).

(ii) P_{30} corresponds to 30% of the total frequency.

$\frac{30}{100} \times 160 = 48$.

The estimated value of P_{30} is 8 minutes (to the nearest minute).

(iii) P_{50} corresponds to 50% of the total frequency.

$\frac{50}{100} \times 160 = 80$.

The estimated value of P_{50} is 14 minutes (to the nearest minute).

(iv) P_{90} corresponds to 90% of the total frequency.

$\frac{90}{100} \times 160 = 144$.

The estimated value of P_{90} is 36 minutes (to the nearest minute).

b) P_{50} is equivalent to the median and Q_2.

c) The value of 14 minutes for P_{50} means that half of the students took 14 minutes or less on average to get school every morning.

d) 30 minutes on the horizontal axis corresponds to the value of 130 on the vertical axis. $\frac{130}{160} \times 100 = 81.25$, so this student falls in the 81st percentile (rounded off to the nearest percentile).

Assess your progress

1. **a)** $\overline{x} = \frac{142 + 153 + 181 + 183 + 126}{5} = 157$ km

b)

Distance (x)	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
142	157	–15	225
153	157	–4	16
181	157	24	576
183	157	26	676
126	157	–31	961

$\Sigma(x - \overline{x})^2 = 225 + 16 + 576 + 676 + 961 = 2\,454$

$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2}{n} = \frac{2\,454}{5} = 490.8$

c) $\Sigma = \sqrt{490.8} = 22.15$ km

d) $\overline{x} - \Sigma = 157 - 22.15 = 134.85$ km
$\overline{x} + \Sigma = 157 + 22.15 = 179.15$ km
$\therefore$ 126 km, 181 km and 183 km do not fall within one standard deviation from the mean.

2. **a)** First calculate the midpoints of the classes.

Capacity (ℓ)	5–9	10–14	15–19	20–24	25–29
Midpoint	7	12	17	22	27
Frequency	9	12	16	9	4

Estimated sum of all the data values: (7×9)
$+ (12 \times 12) + (17 \times 16) + (22 \times 9) + (27 \times 4) = 785$.
There are $(9 + 12 + 16 + 9 + 4) = 50$ data values.
The estimated mean $= \frac{785}{50} = 15.7$ ℓ

b)

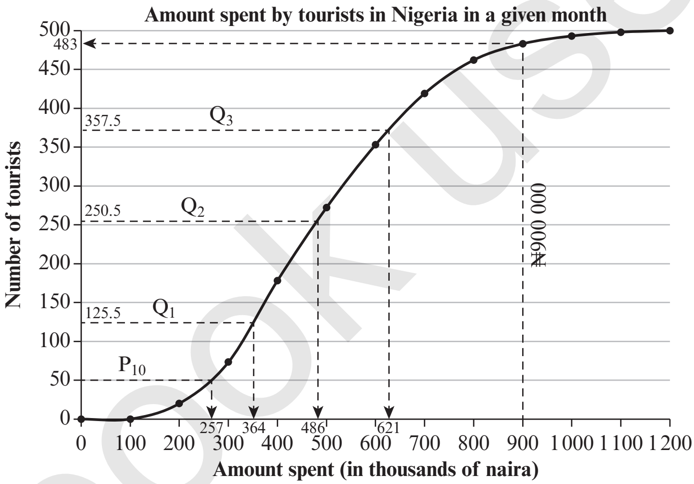
Capacity (ℓ)	Midpoint (x)	Frequency	Mean ($\overline{x}$)	Deviation ($x - \overline{x}$)	Square of deviation $(x - \overline{x})^2$
5–9	7	9	15.7	–8.7	75.69
10–14	12	12	15.7	–3.7	13.69
15–19	17	16	15.7	1.3	1.69
20–24	22	9	15.7	6.3	39.69
25–29	27	4	15.7	11.3	127.69

(i) $\Sigma[(x - \overline{x})^2 \times f] = (75.69 \times 9) + (13.69 \times 12)$
$+ (1.69 \times 16) + (39.69 \times 9) + (127.69 \times 4)$
$= 5\,043.65$

$$\therefore \Sigma^2 = \frac{\Sigma(x - \overline{x})^2 \times f}{n}$$
$$= \frac{5\,043.65}{50}$$
$$= 100.87 \text{ (correct to two decimal places)}$$

(ii) $\Sigma = \sqrt{\Sigma^2}$
$= \sqrt{100.87}$
$= 10.04$ (correct to two decimal places)

3. a)



(i) There are 500 data values. Q_2 lies between the 250th and the 251st data values (i.e. 250.5). The estimated value of Q_2 is ₦486 000.

(ii) Q_1 lies between the 125th and the 126th data values (i.e. 125.5). The estimated value of Q_1 is ₦364 000.

(iii) Q_3 lies between the 375th and the 376th data values (i.e. 375.5). The estimated value of Q_3 is ₦621 000.

(iv) P_{10} corresponds to 10% of the total frequency.
$\frac{10}{100} \times 500 = 50$
The estimated value of P_{10} is ₦2 570 000.

b) ₦9 000 000 on the horizontal axis corresponds to the value of 483 on the vertical axis. $\frac{483}{500} \times 100 = 96.6$, so this tourist falls in the 97th percentile (rounded off to the nearest percentile).

TOPIC 17

Probability

Introduction

In JSS1 and JSS2 your students were introduced to the concept of probability. In this topic they will build on this knowledge as they investigate experimental probability, learn about complementary, mutually exclusive and independent events, perform experiments with or without replacement and solve various practical problems involving probability.

Common difficulties in this topic

Students usually enjoy working with probability, as this work provides a pleasant change of pace from 'normal' mathematics. They can relate to the idea of chance and usually find it interesting to calculate the probabilities of different events. However, the ideas of complementary, independent and mutually exclusive events can be confusing.

It will be a great help if you prepare a chart as described in the next section and display it in your classroom. Also when solving problems that involve performing experiments, students need to think carefully about each context and decide whether or not the experiment is done with or without replacement.

Preparation

Prepare a chart that summarises all the different concepts and terminology that have been introduced in this topic (with examples), as well as a chart that shows the definitions of complementary, mutually exclusive and independent events.

Make sure that you have the necessary equipment for your students to use when performing the practical experiments in this topic. They will need coins and dice (one of each for every two students).

Try to have a few packs of playing cards available for your students to handle. This will help those students who are unfamiliar with cards and card games.

Make a loaded die using a piece of chalk and allow your students to compare this to a normal, fair die.

Have examples of stock market reports, annual share reports and any statistics about the capital market. You can find these in newspapers or in financial magazines. Again, allow your students to browse through these to familiarise themselves with example of statistics in the real world.

Introduction for students

Draw a probability scale on the board and mark it with intervals of tenths, from 0 to 1. Ask a volunteer from your class to write the corresponding decimals below each fraction. A second volunteer should write the corresponding percentages below the decimals.

Give your class examples of everyday events and ask them where each one would be placed on the probability scale. For example, the event 'it will be dark at midnight tonight' would rate as 1 or 100% on the probability scale. On the other hand, the event 'your cat will turn into a dog' would rate as 0 on the probability scale. Ask your class to suggest events that would fall between 0 and 1, and let them indicate where each event would fall on the probability scale.

Answers

Exercise 17.1

Students work in pairs for this exercise.

1. **a)** There are 2 different possible outcomes.
b) 5 trials took place.
c) $\frac{1}{2}$ **d)** $\frac{3}{5}$

2. **a)** There are 6 different possible outcomes.
b) 10 trials took place.
c) $\frac{1}{6}$ **d)** $\frac{3}{10}$ **e)** $\frac{1}{6}$
f) 0 **g)** $\frac{2}{6} = \frac{1}{3}$ **h)** $\frac{6}{10} = \frac{3}{5}$

3. **a)** There are 52 different possible outcomes.
b) 12 trials took place.
c) $\frac{26}{52} = 0.50$ **d)** $\frac{8}{12} = 0.67$ **e)** $\frac{13}{52} = 0.25$

f) $\frac{1}{12} = 0.08$ g) $\frac{10}{13} = 0.77$ h) $\frac{9}{12} = 0.75$

i) $\frac{12}{52} = 0.23$ j) $\frac{3}{12} = 0.25$ k) $\frac{1}{52} = 0.02$

l) $\frac{2}{12} = 0.17$ m) $\frac{4}{52} = 0.08$ n) 0.00

Exercise 17.2

1. **a)**

		Second die					
		1	2	3	4	5	6
First die	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

b) All the outcomes of which the numbers on both dice are the same are on the main diagonal from top left to bottom right.

c) All the outcomes of which the sum of the numbers on the dice is 7 are on the main diagonal from bottom left to top right.

2. {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

3.

		Second ball				
		R	Y	O	G	B
First ball	R	R, R	R, Y	R, O	R, G	R, B
	Y	Y, R	Y, Y	Y, O	Y, G	Y, B
	O	O, R	O, Y	O, O	O, G	O, B
	G	G, R	G, Y	G, O	G, G	G, B
	B	B, R	B, Y	B, O	B, G	B, B

4. {OC, OM, OJ, PC, PM, PJ}

5. {(₦2, ₦1), (₦2, 50K), (₦1, ₦1), (₦1, 50K)}

Exercise 17.3

1. Students' own answers. They should see that average number of heads per toss more closely approaches the theoretical probability of getting heads ($\frac{1}{2}$ or 0.5) as the number of trials increases.

2. Students' own answers. They should see that the average of the number of 6s more closely approaches the theoretical probability of getting a 6 ($\frac{1}{6}$ or $0.1\dot{6}$) as the number of trials increases.

Exercise 17.4

1. **a)** $1 - 0.74 = 0.26$ **b)** $1 - \frac{1}{4} = \frac{3}{4}$
   c) $1 - \frac{11}{50} = \frac{39}{50}$ **d)** $1 - 1 = 0$

2. **a)** $1 - 0 = 1$ **b)** $1 - \frac{6}{25} = \frac{19}{25}$
   c) $1 - \frac{1}{10} = \frac{9}{10}$ **d)** $1 - 0.59 = 0.41$

3. **a)** P(an Ace) $= \frac{4}{52} = \frac{1}{13}$
   b) P(not an Ace) $= 1 - \frac{1}{13} = \frac{12}{13}$
   c) P(a black court card) $= \frac{6}{52} = \frac{3}{26}$
   d) P(not a black court card) $= 1 - \frac{3}{26} = \frac{23}{26}$

4. **a)** P(an odd number) $= \frac{3}{6} = \frac{1}{2}$
   b) P(a number less than 4) $= \frac{3}{6} = \frac{1}{2}$
   c) Even through the sum of the probabilities is 1, they are not complementary events.
   The complementary event of (getting an odd number) is (not getting an odd number) OR (getting an even number).
   The complementary event of (getting a number less than 4) is (not getting a number less than 4) OR (getting a number greater than or equal to 4) OR (getting a number greater than 3).

Exercise 17.5

1. **a)** Events A and B are mutually exclusive.
   b) Events A and B are not mutually exclusive.
   c) Events A and B are mutually exclusive.
   d) Events A and B are not mutually exclusive.

2. **a)** P(they will go to Enugu or Makurdi) = 15% + 25%
= 40%.

b) P(they will go to Beli or Lokoja) = 20% + 40% = 60%.

c) P(they will not go to Enugu) = 1 − 15% = 85%.

3. **a)** The events (Bisola wins the prize) and (Kole wins the prize) are mutually exclusive because they cannot happen at the same time. Both of them cannot win the prize.

b) **(i)** P(Bisola will win the prize) = $\frac{25}{4\,000}$ = 0.00625.

(ii) P(Kole will win the prize) = $\frac{40}{4\,000}$ = 0.01.

(iii) P(Bisola or Kole will win the prize) = $\frac{65}{4\,000}$
= 0.01625.

(iv) P(neither Bisola nor Kole will win the prize)
= 1 − 0.01625 = 0.98375.

4. **a)** P(the name starts with an A) = $\frac{4}{8} = \frac{1}{2}$

b) P(the name starts with an F) = $\frac{2}{8} = \frac{1}{4}$

c) P(the name starts with an M) = $\frac{2}{8} = \frac{1}{4}$

d) P(the name starts with an A or an F) = $\frac{6}{8} = \frac{3}{4}$

e) P(the name starts with an F or an M) = $\frac{4}{8} = \frac{1}{2}$

5. **a)** The number of days in the month can be 28, 29, 30 or 31. If these relative frequencies are accurate and have not been rounded off, then each relative frequency multiplied by the days of the month must give a whole number, so we try them one by one. Using the first relative frequency:

$0.0\dot{3} \times 28 = 0.93333\ldots$

$0.0\dot{3} \times 29 = 0.96666\ldots$

$0.0\dot{3} \times 30 = 1$

$0.0\dot{3} \times 31 = 1.03333\ldots$

∴ There were 30 days in that month.

b) $0.0_3 + 0.1 + 0.1\dot{6} + 0.3 + 0.2 + 0.1\dot{3} + 0.0\dot{6} = 1$

c) **(i)** P(less than 31 °C) = $0.0\dot{3} + 0.1 + 0.1\dot{6} + 0.3 = 0.6$

(ii) P(greater than 27 °C) = $1 - 0.0\dot{3} = 0.9\dot{6}$

(iii) P(29 °C or 32 °C) = $0.1\dot{6} + 0.1\dot{3} = 0.3$

d) The average maximum daily temperature for that month
= $0.0\dot{3}$ × 27 °C + 0.1 × 28 °C + $0.1\dot{6}$ × 29 °C + 0.3 × 30 °C + 0.2 × 31 °C + $0.1\dot{3}$ × 32 °C + $0.0\dot{6}$ × 33 °C
= 30.2 °C

Exercise 17.6

1. **a)** Events A and B are independent.
b) Events A and B are not independent.
c) Events A and B are not independent.
d) Events A and B are not independent.
e) Events A and B are not independent.
f) Events A and B are independent.

2. **a)** P(heads, an odd number and a red card) $= \frac{1}{2} \times \frac{3}{6} \times \frac{26}{52}$
$= \frac{78}{624} = \frac{1}{8}$
b) P(heads, a number greater than 4 and a court card)
$= \frac{1}{2} \times \frac{2}{6} \times \frac{12}{52} = \frac{24}{624} = \frac{1}{26}$
c) P(tails, a five and an Ace) $= \frac{1}{2} \times \frac{1}{6} \times \frac{4}{52} = \frac{4}{624} = \frac{1}{156}$

3. **a)** 70, 74, 77, 80, 84, 87, 90, 94, 97
b) **(i)** P(even) $= \frac{6}{9} = \frac{2}{3}$
(ii) P(multiple of 5) $= \frac{3}{9} = \frac{1}{3}$
(iii) P(multiple of 11) $= \frac{1}{9}$
(iv) P(greater than 80) $= \frac{5}{9}$
(v) P(less than 79) $= \frac{3}{9} = \frac{1}{3}$

Exercise 17.7

1. **a)** If a red ball is not chosen, this means that a green ball is chosen, and vice versa, ∴ B is the complement of A and A is the complement of B.
b) P(choosing a green ball) $= 1 - \frac{5}{7} = \frac{2}{7}$
c) Let the number of balls in the bag be x.
$\therefore \frac{2}{7} \times x = 8$
$\therefore 2x = 56$
$\therefore x = 28$
∴ there are 28 balls altogether.

2. **a)** P(A) × P(B) = 0.6 × 0.3 = 0.18
∴ P(A and B) = P(A) × P(B)
∴ events A and B are independent.
b) P(A) × P(B) = 0.2 × 0.9 = 0.18
∴ P(A and B) ≠ P(A) × P(B)
∴ events A and B are not independent.
c) P(A) + P(B) = 0.1 + 0.75 = 0.85
∴ P(A or B) ≠ P(A) + P(B)
∴ events A and B are not mutually exclusive.

d) P(A) + P(B) = 0.45 + 0.25 = 0.7
$\therefore$ P(A or B) = P(A) + P(B)
$\therefore$ events A and B are mutually exclusive.

3. **a)** P(three 6s) = $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$
b) P(anything other than three 6s) = $1 - \frac{1}{216} = \frac{215}{216}$

4. **a)** The events (choosing a ten's digit) and (choosing a unit's digit) are independent. They are chosen separately and do not influence one another.
b) $3 \times 3 = 9$
$\therefore$ 9 different two-digit numbers can be formed in this way.
c) **(i)** P(even) = P(the two-digit number must end in a 0 or a 2) = $\frac{2}{3}$
(ii) P(odd) = P(the two-digit number must end in a 1) = $\frac{1}{3}$
(iii) The only possible prime numbers are 21, 31 and 41. Of these, 31 and 41 are prime, $\therefore$ P(prime) = $\frac{2}{9}$
(iv) P(multiple of five) = P(the two-digit number must end in a 0) = $\frac{1}{3}$

5. **a)** $26 \times 10 = 260$, $\therefore$ 260 different pin codes can be formed in this way.
b) P(vowel) = $\frac{5}{26}$
P(even number) = $\frac{1}{2}$
$\therefore$ P(a vowel followed by an even number) = $\frac{5}{26} \times \frac{1}{2} = \frac{5}{52}$

6. **a)** P(all three players score) = $\frac{3}{4} \times \frac{2}{3} \times \frac{3}{5} = \frac{3}{10}$
b) P(Dele scores but Segun does not) = $\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$
c) P(only Yomi scores) = $\frac{3}{4} \times \frac{1}{3} \times \frac{2}{5} = \frac{1}{10}$
d) P(none of them scores) = $\frac{1}{4} \times \frac{1}{3} \times \frac{2}{5} = \frac{1}{30}$

7. **a)** A and B are not mutually exclusive events, because it is possible for both of them to happen at the same time. (We know this because P(A and B) = $\frac{3}{20}$.)
b) P(A) × P(B) = $\frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$ and P(A and B) = $\frac{3}{20}$
$\therefore$ A and B are independent events.
c) **(i)** P(rice will be served, but beans will not)
$= \frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$.
(ii) P(beans will be served, but rice will not)
$= \frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10}$.

(iii) P(neither rice nor beans will be served)
$= \frac{2}{5} \times \frac{3}{4} = \frac{6}{20} = \frac{3}{10}$.

8. a) (i) Events A and B are mutually exclusive, because they cannot happen at the same time. If Bolaji wins, then Ikeade loses, and vice versa.

(ii) Events A and B are independent. If Sade wins, this will not influence Habib, and vice versa.

b) (i) P(Bolaji, Sade and Habib all win)
$= \frac{1}{4} \times \frac{2}{3} \times \frac{3}{10} = \frac{6}{120} = \frac{1}{20}$

(ii) P(Bolaji, Sade and Habib all lose)
$= \frac{3}{4} \times \frac{1}{3} \times \frac{7}{10} = \frac{7}{40}$

(iii) P(Bolaji and Sade win, but Habib loses)
$= \frac{1}{4} \times \frac{2}{3} \times \frac{7}{10} = \frac{7}{60}$

Exercise 17.8

1. a) (i) {RR, RB, RG, RY, BR, BB, BG, BY, GR, GB, GG, GY, YR, YB, YG, YY}

(ii) 16 outcomes

b) (i) {RB, RG, RY, BR, BG, BY, GR, GB, GY, YR, YB, YG}

(ii) 12 outcomes

c) (i) {RBG, RBY, RGB, RGY, RYB, RYG, BRG, BRY, BGR, BGY, BYR, BYG, GRB, GRY, GBR, GBY, GYR, GYB, YRB, YRG, YBR, YBG, YGR, YGB}

(ii) 24 outcomes

2. a) (i) P(RRR) = 0

P(BBB) $= \frac{4}{11} \times \frac{4}{11} \times \frac{4}{11} = \frac{64}{1\,331}$

P(WWW) $= \frac{5}{11} \times \frac{5}{11} \times \frac{5}{11} = \frac{125}{1\,331}$

$\therefore$ P(three balls of the same colour) = P(RRR) + P(BBB) + P(WWW) $= 0 + \frac{64}{1\,331} + \frac{125}{1\,331} = \frac{189}{1\,331}$

(ii) P(RBW) $= \frac{2}{11} \times \frac{4}{11} \times \frac{5}{11} = \frac{40}{1\,331}$

P(RWB) $= \frac{2}{11} \times \frac{5}{11} \times \frac{4}{11} = \frac{40}{1\,331}$

P(BRW) $= \frac{4}{11} \times \frac{2}{11} \times \frac{5}{11} = \frac{40}{1\,331}$

P(BWR) $= \frac{4}{11} \times \frac{5}{11} \times \frac{2}{11} = \frac{40}{1\,331}$

P(WRB) $= \frac{5}{11} \times \frac{2}{11} \times \frac{4}{11} = \frac{40}{1\,331}$

P(WBR) $= \frac{5}{11} \times \frac{4}{11} \times \frac{2}{11} = \frac{40}{1\,331}$

$\therefore$ P(three balls of different colours) = P(RBW) + P(RWB) + P(BRW) + P(BWR) + P(WRB) + P(WBR) $= \frac{40}{1\,331} + \frac{40}{1\,331} + \frac{40}{1\,331} + \frac{40}{1\,331} + \frac{40}{1\,331} + \frac{40}{1\,331}$
$= \frac{240}{1\,331}$

(iii) P(at least one red ball) = 1 − P(no red balls)

P(BBB) $= \frac{4}{11} \times \frac{4}{11} \times \frac{4}{11} = \frac{64}{1\,331}$

P(BBW) $= \frac{4}{11} \times \frac{4}{11} \times \frac{5}{11} = \frac{80}{1\,331}$

P(BWB) $= \frac{4}{11} \times \frac{5}{11} \times \frac{4}{11} = \frac{80}{1\,331}$

P(BWW) $= \frac{4}{11} \times \frac{5}{11} \times \frac{5}{11} = \frac{100}{1\,331}$

P(WBB) $= \frac{5}{11} \times \frac{4}{11} \times \frac{4}{11} = \frac{80}{1\,331}$

P(WBW) $= \frac{5}{11} \times \frac{4}{11} \times \frac{5}{11} = \frac{100}{1\,331}$

P(WWB) $= \frac{5}{11} \times \frac{5}{11} \times \frac{4}{11} = \frac{100}{1\,331}$

P(WWW) $= \frac{5}{11} \times \frac{5}{11} \times \frac{5}{11} = \frac{125}{1\,331}$

P(no red balls) = P(BBB) + P(BBW) + P(BWB) + P(BWW) + P(WBB) + P(WBW) + P(WWB) + P(WWW) $= \frac{64}{1\,331} + \frac{80}{1\,331} + \frac{80}{1\,331} + \frac{100}{1\,331} + \frac{80}{1\,331} + \frac{100}{1\,331} + \frac{100}{1\,331} + \frac{125}{1\,331} = \frac{729}{1\,331}$

$\therefore$ P(at least one red ball) = 1 − P(no red balls)
$= 1 - \frac{729}{1\,331} = \frac{602}{1\,331}$

b) (i) P(RRR) = 0

P(BBB) $= \frac{4}{11} \times \frac{3}{10} \times \frac{2}{9} = \frac{24}{990}$

P(WWW) $= \frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} = \frac{60}{990}$

$\therefore$ P(three balls of the same colour) = P(RRR) + P(BBB) + P(WWW) $= 0 + \frac{24}{990} + \frac{60}{990} = \frac{84}{990} = \frac{14}{165}$

(ii) P(RBW) $= \frac{2}{11} \times \frac{4}{10} \times \frac{5}{9} = \frac{40}{990}$

P(RWB) $= \frac{2}{11} \times \frac{5}{10} \times \frac{4}{9} = \frac{40}{990}$

P(BRW) $= \frac{4}{11} \times \frac{2}{10} \times \frac{5}{9} = \frac{40}{990}$

P(BWR) $= \frac{4}{11} \times \frac{5}{10} \times \frac{2}{9} = \frac{40}{990}$

P(WRB) $= \frac{5}{11} \times \frac{2}{10} \times \frac{4}{9} = \frac{40}{990}$

P(WBR) $= \frac{5}{11} \times \frac{4}{10} \times \frac{2}{9} = \frac{40}{990}$

$\therefore$ P(three balls of different colours) = P(RBW) + P(RWB) + P(BRW) + P(BWR) + P(WRB) + P(WBR) $= \frac{40}{990} + \frac{40}{990} + \frac{40}{990} + \frac{40}{990} + \frac{40}{990} + \frac{40}{990}$
$= \frac{240}{990} = \frac{8}{33}$

(iii) P(at least one red ball) = 1 − P(no red balls)

$P(BBB) = \frac{4}{11} \times \frac{3}{10} \times \frac{2}{9} = \frac{24}{990}$

$P(BBW) = \frac{4}{11} \times \frac{3}{10} \times \frac{5}{9} = \frac{60}{990}$

$P(BWB) = \frac{4}{11} \times \frac{5}{10} \times \frac{3}{9} = \frac{60}{990}$

$P(BWW) = \frac{4}{11} \times \frac{5}{10} \times \frac{4}{9} = \frac{80}{990}$

$P(WBB) = \frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} = \frac{60}{990}$

$P(WBW) = \frac{5}{11} \times \frac{4}{10} \times \frac{4}{9} = \frac{80}{990}$

$P(WWB) = \frac{5}{11} \times \frac{4}{10} \times \frac{4}{9} = \frac{80}{990}$

$P(WWW) = \frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} = \frac{60}{990}$

P(no red balls) = P(BBB) + P(BBW) + P(BWB) + P(BWW) + P(WBB) + P(WBW) + P(WWB) + P(WWW) $= \frac{24}{990} + \frac{60}{990} + \frac{60}{990} + \frac{80}{990} + \frac{60}{990} + \frac{80}{990} + \frac{80}{990} + \frac{60}{990} = \frac{504}{990} = \frac{28}{55}$

∴ P(at least one red ball) = 1 − P(no red balls) $= 1 - \frac{28}{55} = \frac{27}{55}$

3. a) 26 × 26 × 26 × 10 × 10 × 10 = 17 576 000, ∴ he can make 17 576 000 different passwords.

b) 1 × 26 × 26 × 10 × 10 × 1 = 67 600, ∴ he can make 67 600 different passwords.

P(password starts with an A and ends with a 9) $= \frac{67\,600}{17\,576\,000} = \frac{1}{260}$

c) 26 × 25 × 24 × 10 × 9 × 8 = 11 232 000, ∴ he can make 11 232 000 different passwords.

d) 1 × 25 × 24 × 9 × 8 × 1 = 43 200, ∴ he can make 43 200 different passwords.

P(password starts with an A and ends with a 9) $= \frac{43\,200}{11\,232\,000} = \frac{1}{260}$

Exercise 17.9

1. a) The infant mortality rates decreased steadily from 2010 to 2013.

b) P(a baby born alive in 2011 did not survive) $= \frac{79}{1\,000}$

c) A couple had a baby in 2010 and another in 2013.

(i) P(both babies were born alive, but did not survive) $= \frac{82}{1\,000} \times \frac{74}{1\,000} = \frac{6\,068}{1\,000\,000} = \frac{1\,517}{250\,000}$

(ii) P(both babies were born alive and survived)
$= \left(1 - \frac{82}{1\,000}\right) \times \left(1 - \frac{74}{1\,000}\right) = \frac{918}{1\,000} \times \frac{926}{1\,000}$
$= \frac{850\,068}{1\,000\,000} = \frac{212\,517}{250\,000}$

2. a) (i) P(high dividends from both shares)
= 0.4 × 0.2 = 0.08
(ii) P(medium dividends from both shares)
= 0.1 × 0.4 = 0.04
(iii) P(low dividends from both shares)
= 0.3 × 0.3 = 0.09
(iv) P(no dividends from both shares)
= 0.2 × 0.1 = 0.02
b) 0.08 + 0.04 + 0.09 + 0.02 = 0.23
All of the probabilities on question a) are based on the fact that the shares perform identically. None of the combinations in which they perform differently are included.

3. a) M and W are complementary events, because
P(M) + P(W) = 1
M and W are mutually exclusive events, because
P(M and W) = 0
M and W are not independent events, because
P(M and W) ≠ P(M) × P(W)
b) P(a woman's name was drawn) = $\frac{5}{12}$ = 0.42

4. a) The size of the total population is 820 people.
b) (i) P(a man) = $\frac{162}{820}$ = 0.20
(ii) P(a girl) = $\frac{214}{820}$ = 0.26
(iii) P(an adult) = 0.42
(iv) P(a female) = 0.49

5. a)

	Section A	Section B	Section C	Section D	Total
2011	21	39	10	4	74
2012	23	37	12	8	80
2013	19	41	11	6	77
Total	63	117	33	18	231

b) Section A: $\frac{63}{231} = 27.3\%$

Section B: $\frac{117}{231} = 50.6\%$

Section C: $\frac{33}{231} = 14.3\%$

Section D: $\frac{18}{231} = 7.8\%$

c) Section B, because more than half of all the accidents that occurred on the pass over the three-year period occurred in that section.

6. **a)** Damola: 0.40, Wole: 0.33, Gbenga: 0.37

b) Damola

c) The team manager should definitely also take into account the number of penalties that each goalkeeper faced. This will give him a much better idea of how each goalkeeper performed.

d) Damola: $\frac{2}{5} \times 150 = 60$ penalties

Wole: $\frac{1}{3} \times 150 = 50$ penalties

Gbenga: $\frac{11}{30} \times 150 = 55$ penalties

7. **a)** $1 - 8.5\% = 0.915$

P(woman actually has breast cancer) $= 0.05 \times 0.915$

$= 0.04575$

b) **(i)** $0.05 \times 3\,000 = 150$ women

(ii) $0.04575 \times 3\,000 \approx 137$ women

8. **a)** **(i)** P(a student had visited a neighbouring country) $= \frac{55}{250} = \frac{11}{50}$

(ii) P(a student was planning to study Science at a tertiary level) $= \frac{31}{250}$

(iii) P(a student expected to be married before the age of 30) $= \frac{152}{250} = \frac{76}{125}$

(iv) P(a student hoped to have at least three children) $= \frac{198}{250} = \frac{99}{125}$

b) **(i)** $\frac{11}{50} \times 40\,000 = 8\,800$ students

(ii) $\frac{31}{250} \times 40\,000 = 4\,960$ students

(iii) $\frac{76}{125} \times 40\,000 = 24\,320$ students

(iv) $\frac{99}{125} \times 40\,000 = 31\,680$ students

Assess your progress

1. **a)** Yes. P(A) = P(B) = P(C) = P(D) = $\frac{4}{8} = \frac{1}{2}$

b) Events A and B are complementary, because B = 'not A' and vice versa.

c) Events A and B are mutually exclusive, because P(A and B) = 0.

d) C and D = {13, 21}

$\therefore$ P(C and D) = $\frac{2}{8} = \frac{1}{4}$

and P(C) × P(D) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore$ P(C and D) = P(C) × P(D)

$\therefore$ C and D are independent events.

2. **a)** {1R, 1B, 1P, 1O, 2R, 2B, 2P, 2O, 3R, 3B, 3P, 3O, 4R, 4B, 4P, 4O, 5R, 5B, 5P, 5O}

b) **(i)** P(3 and purple) = $\frac{1}{20}$

(ii) P(an odd number and blue) = $\frac{3}{20}$

(iii) P(an even number and either red or orange) = $\frac{4}{20} = \frac{1}{5}$

3. **a)** There are ten ways of choosing each digit, so 10 × 10 × 10 = 1 000 different pin codes can be formed.

b) There are ten ways of choosing the first digit. There are nine ways of choosing the second digit. There are eight ways of choosing the third digit.

$\therefore$ 10 × 9 × 8 = 720 different pin codes can be formed.

4. P(two socks of the same colour)

= P(black, black) + P(blue, blue) + P(brown, brown)

$= \left(\frac{3}{12} \times \frac{2}{11}\right) + \left(\frac{4}{12} \times \frac{3}{11}\right) + \left(\frac{5}{12} \times \frac{4}{11}\right) = \frac{38}{132} = \frac{19}{66}$

5. **a)** P(both of them chose cooking) = $\frac{20}{200} \times \frac{19}{199} = \frac{19}{1\,990}$

b) P(neither of them chose music) = $\frac{155}{200} \times \frac{154}{199} = \frac{2\,387}{3\,980}$

TOPIC 18 Revision and exam practice

1 Number and numeration

Exercise 18.1

1. D	**2.** C	**3.** C	**4.** A	**5.** B
6. C	**7.** A	**8.** B	**9.** D	**10.** B
11. A	**12.** B	**13.** C	**14.** A	**15.** D
16. B	**17.** D	**18.** A	**19.** B	**20.** C
21. B	**22.** A	**23.** C	**24.** C	**25.** B

Exercise 18.2

1. **a)** 561 **b)** 512 **c)** 337

2. **a)** $33\ 122_4$ **b)** $12\ 421_5$ **c)** $1\ 732_8$ **d)** $3DA_{16}$

3. **a)** 5 010 000 **b)** 0.000000123

4. **a)** 8.031×10^{-5} **b)** 1.00304×10^8

5. **a)** 5.82×10^9 **b)** 2.28×10^{-6}

6. **a)** 2.14×10^3 **b)** 6.96×10^6

7. **a)** 3.981×10^6 **b)** -4.848×10^{-4}

8. **a)** 12^{12} **b)** 7^{-3} **c)** 11^5 **d)** 10^{10}

9. **a)** $r = 2$ **b)** $T_1 = 3$ **c)** $S_5 = 93$

10. 25, 16, 9

11. **a)** $T_n = 4n - 5$ **b)** $T_n = 3n - 25$

12. $T_{10} = 14$

13. $-\frac{5}{4}, -\frac{5}{8}, -\frac{5}{16}$

14. **a)** $T_n = \frac{1}{2}(-2)^{n-1}$ or $\frac{-1}{4}(-2)^n$

b) $T_n = 6\left(\frac{1}{3}\right)^{n-1}$ or $18\left(\frac{1}{3}\right)^n$

15. **a)** {2} **b)** {1, 3, 7, 9, 13, 15}

c) {1, 5, 6, 9, 11, 12, 15} **d)** {4, 5, 6, 8, 10, 11, 12, 14}

2 Algebraic processes

Exercise 18.3

1. C	**2.** A	**3.** C	**4.** D	**5.** D
6. A	**7.** B	**8.** A	**9.** A	**10.** C

Exercise 18.4

1. −2 and −7

2. $x = -\frac{1}{5}$ or 1 and $y = -\frac{8}{5}$ or 2

3. **a)** $(x - 6)^2$ **b)** $(x - 8)(x - 4)$
c) $3(x + 3)(x + 2)$ **d)** $4(x + 5)(x - 5)$

4. **a)** $x = -8$ or $x = 5$ **b)** $x = 1$ or $x = -\frac{1}{3}$
c) $x = \pm 3$ **d)** $x = 0$ or $x = 1$

5. **a)** $x = 1.85$ or $x = -1.35$ **b)** $x = 0.61$ or $x = -4.11$
c) $x = -0.17$ or $x = -1.69$ **d)** $x = 2.26$ or $x = -0.59$

6. **a)** $\frac{-5 \pm 2\sqrt{10}}{3}$ **b)** $\frac{1 \pm \sqrt{61}}{10}$ **c)** $\frac{5 \pm \sqrt{-47}}{4}$ **d)** $\frac{-5 \pm \sqrt{73}}{6}$

7. **a)** $p = -5$ **b)** The roots are $\frac{5}{2}$ and $-\frac{1}{2}$.

8. **a)** (−1; 0) and (3; 0) **b)** $m = 2$ and $k = 2$
c) B(−1; 0) and E(1; 4) **d)** $x = 1$

9. 27 and 7

10. ₦600 at 3% and ₦1 800 at 5%

3 Geometry

Exercise 18.5

1. C	**2.** D	**3.** D	**4.** B	**5.** A
6. A	**7.** D	**8.** C	**9.** B	**10.** C
11. D	**12.** B	**13.** A	**14.** A	**15.** D
16. C	**17.** B	**18.** A	**19.** B	**20.** A

Exercise 18.6

1. **a)** $\hat{R}_1 = 90°$ (∠ in a semicircle)
b) $\hat{Q}_1 = 50°$ (sum of ∠s in a △ = 180°)
c) $\hat{A} = 130°$ (opp. ∠s cyclic quad)
d) 140° (opp. ∠s cyclic quad)

2. **a)** $A\hat{C}B = 90°$ (∠ in a semicircle)
OE ⊥ BC at D (given)
∴ EO ∥ CA (corr. ∠s equal)

b) $\hat{A} = x$ (tangent PC, chord AC)
$\hat{O}_1 = x$ (corr. ∠s equal)

c) $\hat{B}_1 = 90° - x$ (sum of ∠s in △ABC)
∴ $\hat{B}_2 = 90° - x$ (ABP straight line)
∴ $\hat{P} = 90° - 2x$ (sum of ∠s in △PBC)

3. $6x = 180°$ (∠s on a straight line)
∴ $x = 30°$ and $2x = 60°$
∴ $C\hat{B}D = \hat{E}$
∴ ABC is a tangent to the circle at B.

4. **a)** **(i)** $\hat{C} = 35°$ (tan AE, chord DE)
(ii) $A\hat{E}B = 65°$ (ext. ∠ = sum int. opp. ∠s in △DEC)

b) $A\hat{B}E = 65°$ (AB = AE)
∴ ADEB is a cyclic quad. (ext. ∠ = int. opp. ∠)

c) $A\hat{B}D = 35°$ (equal ∠s in same segment)
∴ $A\hat{B}D = \hat{C}$
∴ AB is a tangent to the circle through B, D and C.

d) $D\hat{B}E = A\hat{B}E - A\hat{B}D = 30°$
∴ $B\hat{D}E = 85°$ (sum of ∠s in △DBE)

5. **a)** $\hat{S}_1 = 70°$ (alt. ∠s ST ∥ QW)
∴ $\hat{V} = 110°$ (opp. ∠s cyclic quad.)

b) $\hat{Q}_1 = 70°$ (tangent PQ, chord SQ)

c) $\hat{Q}_2 = 80°$ (sum ∠s in △SQW)
∴ $\hat{T}_1 = 100°$ (opp. ∠s cyclic quad.)

6. QT = 3 cm (line from centre ⊥ to chord)
∴ OT = 4 cm (Pythagoras)
∴ TS = 7.5 cm (Pythagoras)
∴ PS = 15 cm (line from centre to midpoint of chord)

7. **a)** $C\hat{P}O = 90°$ (given)
$\therefore CP = DP$ (line from centre ⊥ to chord bisects chord)

b) Let $\hat{C}_2 = \hat{B}_1 = x$ (OC = OB, radii)
$\therefore \hat{O}_2 = 180° - 2x$ (sum of ∠s in △COB)
$\therefore C\hat{D}B = 90° - x$ (∠ at centre = $\frac{1}{2}$ ∠ at circumf.)
$\therefore \hat{B}_2 = x$ (sum of ∠s in △PDB)
$\therefore \hat{B}_1 = \hat{B}_2$

c) $\hat{E} = \hat{B}_1 + \hat{B}_2$ (∠s in same segment)
$\therefore \hat{E} = 2\hat{B}_1 = 2x$

d) $\hat{O}_2 = 180° - 2x$ (proved above)
and $\hat{E} = 2x$
∴ OFEC is a cyclic quad. (opp. ∠s suppl.)

8. **a)** $\hat{C}_2 = x$ (given)
$\therefore \hat{D}_2 = x$ (∠s in same segment)
$\therefore \hat{P}_1 = x$ (tan−chord theorem)
$\hat{B}_2 = x$ (DP = PB)
$\therefore \hat{C}_1 = x$ (∠s in same segment)
$\therefore \hat{P}_4 = x$ (tan−chord theorem)

b) $D\hat{P}B = D\hat{C}B$ (given)
$\therefore 180° - 2x = 2x$ ($\hat{C}_1 = \hat{C}_2 = \hat{P}_1 = \hat{P}_4 = x$)
$\therefore 2x = 90°$ (suppl. ∠s on straight line TPA)
$\therefore x = 45°$

c) $D\hat{P}B = 90° = D\hat{C}B$ (proved above)
∴ DB is a diameter of the given circle

9. **a)** $\hat{A}_1 = x$ (given)
AD and AB are tangents to the circle (AD = BD)
$\hat{B}_1 = x$ (tan−chord theorem)
$\hat{A}_2 = x$ (FA = FB)
$\hat{C} = x$ (corr. ∠s, DC ∥ FB)
$\hat{B}_2 = x$ (DA = DB
$\hat{D}_1 = x$ (alt. ∠s, DC ∥ FB)

b) $\hat{D}_1 = x = \hat{A}_2$ (proved above)
∴ ABED is a cyclic quad. (equal ∠s in same segment)

10. $\hat{R} = 90°$ (∠ in a semicircle)
$\hat{T}_3 = 90°$ (PT ⊥ MN)
∴ TSRN is a cyclic quad. (opp. ∠s suppl.)

11. a) reflex $L\hat{O}N = 260°$ (sum of ∠s around a point)

$L\hat{M}N = 130°$ (∠ at centre = twice ∠ at circumf.)

b) $\hat{N}_1 = \hat{L}_1 = 25°$ (LM = MN, sum of ∠s in isosceles △LMN)

$\hat{N}_1 = \hat{K}$ (∠s in same segment)

$\therefore \hat{K} = 25°$

4 Trigonometry

Exercise 18.7

1. B	**2.** C	**3.** A	**4.** D	**5.** B
6. A	**7.** C	**8.** B	**9.** C	**10.** A
11. D	**12.** D	**13.** C	**14.** B	**15.** D
16. A	**17.** B	**18.** C	**19.** D	**20.** A

Exercise 18.8

1. a) $BD^2 = 13^2 + 16^2 - 2(13)(16) \cos 75 = 317.33$

$\therefore BD = 18$ cm

b) $\cos x = \frac{BD}{AD}$

$\therefore AD = \frac{18}{\cos x}$

$E\hat{A}D = x$ (alt. ∠s, AE ∥ BD)

$\therefore \hat{E} = 180 - 2x$ (∠s in isosceles △)

2. a) $\frac{ZX}{\sin 30} = \frac{3}{\sin 120}$

$\therefore ZX = \frac{3 \sin 30}{\sin 120} = \sqrt{3}$

b) In △XYZ, $XY^2 = (1.5)^2 + (\sqrt{3})^2 - 2(1.5)(\sqrt{3}) \cos 30$

$XY^2 = \frac{3}{4}$

$\therefore XY^2 + ZY^2 = \frac{3}{4} + (1.5)^2 = 3$

and $ZX^2 = 3$

$\therefore$ △XYZ is right angled (Pythagoras)

$\therefore Z\hat{Y}X = 90°$

3. $\frac{BC}{\sin A} = \frac{AB}{\sin C}$

$\therefore BC = \frac{AB \sin A}{\sin C} = \frac{0.67 \sin 37}{\sin 128} = 0.51$ km or 510 m

$\frac{AC}{\sin B} = \frac{AB}{\sin C}$

$\therefore AC = \frac{AB \sin B}{\sin C} = \frac{0.67 \sin 15}{\sin 128} = 0.22$ km or 220 m

4. **a)** $AC^2 = 7.1^2 + 2.8^2 - 2(7.1)(2.8)\cos 58.9$
$= 37.7126\ldots$
$\therefore AC = 6.14$ km

b) $\frac{\sin D\hat{C}A}{2.8} = \frac{\sin 58.9}{6.14}$

$\therefore \sin D\hat{C}A = \frac{\sin 58.9}{6.14} \times 2.8 = 0.3904\ldots$

$\therefore D\hat{C}A = 22.98°$

c) $C\hat{A}B = 22.98°$ (CD ∥ AB)
$\therefore B\hat{C}A = 123.12°$ (sum of ∠s in △ABC)

5. $\frac{KT}{\sin 40} = \frac{5}{\sin 60}$
$\therefore KT = 3.71$ cm
$PT^2 = 7^2 + 5^2 - 2(7)(5)\cos 30 = 13.378\ldots$
$\therefore PT = 3.66$ cm

6. $\hat{C}_1 = 36°$
$G\hat{C}A = 41°$
$GA^2 = (8.3)^2 + (4.8)^2 - 2(8.3)(4.8)\cos 41° = 31.7947\ldots$
$\therefore GA = 5.6$ km

7. $\frac{\sin Q}{91} = \frac{\sin 49.5}{73}$
$\therefore \sin Q = 0.9479\ldots$
$\therefore \hat{Q} = 71.42°$ or $180° - 71.42°$
$\therefore \hat{Q} = 71.42°$ or $108.58°$
If $\hat{Q} = 71.42°$, then $\hat{P} = 59.08°$
$\therefore \frac{QR}{\sin 59.08} = \frac{91}{\sin 71.42}$
$\therefore QR = 82.36$ mm
If $\hat{Q} = 108.58°$, then $\hat{P} = 21.92°$
$\therefore \frac{QR}{\sin 21.92} = \frac{91}{\sin 108.58}$
$\therefore QR = 35.84$ mm

5 Statistics

Exercise 18.9

1. B	**2.** A	**3.** D	**4.** C
5. A	**6.** B	**7.** A	**8** C
9. C	**10.** D	**11.** B	**12.** A
13. B	**14.** D	**15.** C	**16.** D

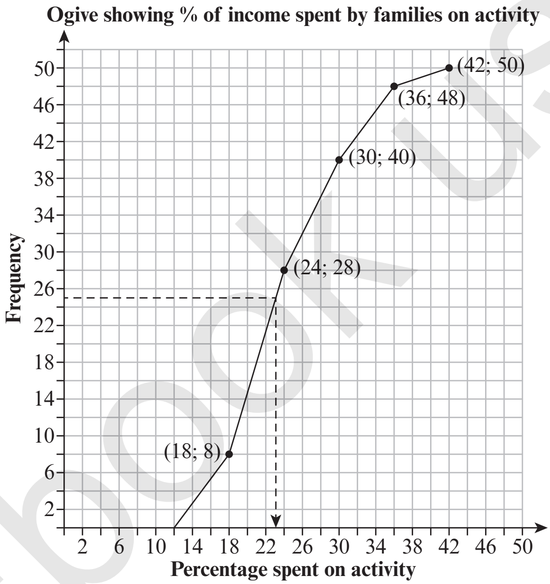
Exercise 18.10

1. **a)**

%	f	Midpoint	cf
$12 < p \leq 18$	8	15	8
$18 < p \leq 24$	20	21	28
$24 < p \leq 30$	12	27	40
$30 < p \leq 36$	8	33	48
$36 < p \leq 42$	2	39	50

b) $\bar{x} = \frac{1\,206}{50} = 24.1$ and $\sigma = 6.4$

c)



2. **a)** median = 51 **b)** $Q_1 = 42$

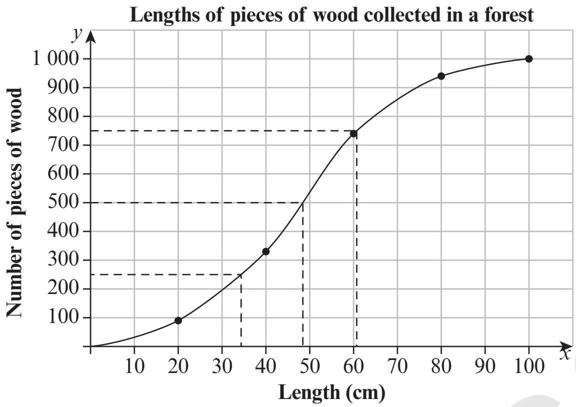
c) mode = 66 **d)** range = 141 − 37 = 104

e) mean = $\frac{448}{7} = 64$ **f)** standard deviation = 33.12

3. **a)**

Length (cm)	Frequency	Cumulative frequency
$l < 20$	90	90
$20 \leq l < 40$	240	330
$20 \leq l < 60$	410	740
$20 \leq l < 80$	200	940
$20 \leq l < 100$	60	1 000

b)



c) median = 48 **d)** IQR = 61 − 35 = 26

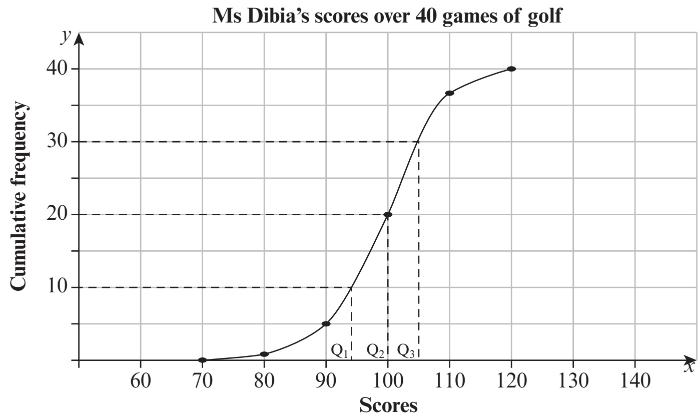
4. **a)** 57 minutes **b)** 72 − 44 = 28 minutes
c) 10%

5. **a)** mean = 62.7 **b)** $\sum(x - \overline{x})^2 = 4\,910.2$
c) standard deviation = 15.67
d) 47.03 ← 62.7 → 78.37
$\therefore \frac{17}{20}$ = 85% scored within one s.d. of the mean.

6. **a)**

Interval	Frequency	Cumulative frequency
$70 < x \leq 80$	1	1
$80 < x \leq 90$	4	5
$90 < x \leq 100$	15	20
$100 < x \leq 110$	17	37
$110 < x \leq 120$	3	40

b)



c) **(i)** 100

(ii) 75th percentile = $Q_3 = 105$

(iii) 25th percentile = $Q_1 = 95$

7. **a)** Approximate mean

$$= \frac{(17 \times 900) + (6 \times 700) + (8 \times 500) + (3 \times 300) + (1 \times 100)}{35}$$

$$= \frac{24\,500}{35}$$

$$= 700$$

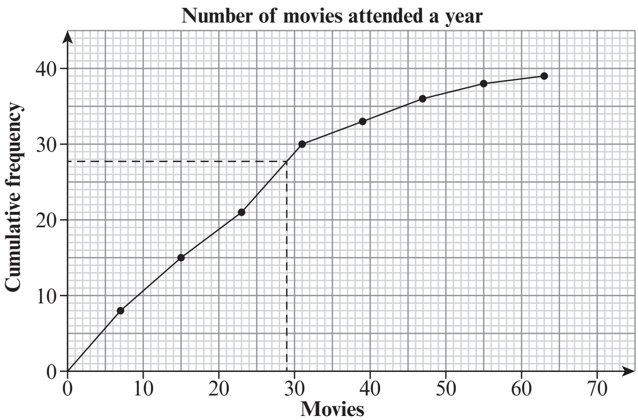
b) **Disagree**: The mean CD4 count for females is higher than the count for males. A lower standard deviation for females means that the results are closer to the higher mean.

OR Agree: The mean CD4 count is very similar for both sexes and the standard deviation values are also similar.

8. **a)**

Number of movies seen	Frequency	Cumulative frequency	Ordered pairs
0–7	8	8	(7; 8)
8–15	7	15	(15; 15)
16–23	6	21	(23; 21)
24–31	9	30	(31; 30)
32–39	3	33	(39; 33)
40–47	3	36	(47; 36)
48–55	2	38	(55; 38)
56–63	1	39	(63; 39)

b)



c) $39 \times \frac{70}{100} = 27.3$

70th percentile = 29 movies

9. a) $\overline{x} = 4.24$

Length (cm)	$(x - \overline{x})$	$(x - \overline{x})^2$
4.3	0.06	0.0036
4.6	0.36	0.1296
5.2	0.96	0.9216
3.4	–0.84	0.7056
4.7	0.46	0.2116
4.3	0.06	0.0036
4.1	–0.14	0.0196
5.0	0.76	0.5776
3.6	–0.64	0.4096
3.2	–1.04	1.0816

b) s.d. = $\sqrt{\frac{4.064}{10}} = 0.64$

10. a) (i) mode = 27

(ii) range = 40 − 11 = 29

b)

Min	Q_1	Q_2	Q_3	Max
11	27	28	32	40

c) IQR = 32 − 27 = 5

d) $\overline{x} = \frac{430}{15} = 28.7$ minutes per call

e)

Calls per day	Frequency	$(x - \overline{x})$	$(x - \overline{x})^2$	$f(x - \overline{x})^2$
11	1	–17.7	313.29	313.29
23	2	–5.7	32.49	64.98
27	4	–1.7	2.89	11.56
28	1	–0.7	0.49	0.49
29	1	0.3	0.09	0.09
32	3	3.3	10.89	32.67
36	2	7.3	53.29	106.58
40	1	11.3	127.69	127.69

f) s.d. = $\sqrt{\frac{657.35}{15}} = 6.6$

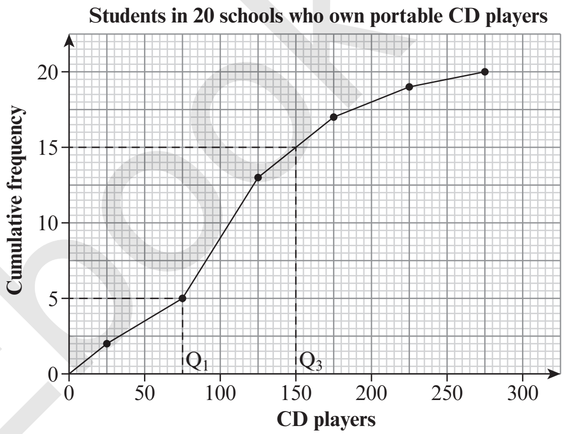
$22.1 \leftarrow 28.7 \rightarrow 35.3$

$\therefore$ 11 calls are within one standard deviation of the mean.

$\therefore \frac{11}{15} \times 100 = 73.3\%$

11. a)

Number of CD players	Number of schools	Cumulative frequency
0–50	2	2
51–100	3	5
101–150	8	13
151–200	4	17
201–250	2	19
251–300	1	20



b) $Q_1 = 75$ and $Q_3 = 150$

c) IQR = 150 − 75 = 75